Publicaciones Matemáticas del Uruguay

Volumen 13, Año 2011

Publicaciones Matemáticas del Uruguay

Published by:
IMERL-Facultad de Ingeniería
CMAT-Facultad de Ciencias
Universidad de la República
http://imerl.fing.edu.uy/pmu

ISSN: 0797-1443

Credits:
Cover design: J. Rodriguez Hertz
LaTeX editor: J. Rodriguez Hertz
using LaTeX's 'confproc' package, version 0.7 (by V. Verfaille)

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Sesiones especiales

PREFACIO

Este volumen contiene el material correspondiente al 3er Coloquio Uruguayo de Matemática, a desarrollarse entre el 20 y el 22 de diciembre en Montevideo, Uruguay.

El mismo contiene las notas de los cuatro cursos a ser dictados durante el transcurso del Coloquio, y los resúmenes de las Sesiones especiales de las diversas áreas.

Agradecemos al Comité Organizador del Coloquio el habernos confiado la edición de las notas correspondientes al mismo. También agradecemos a la organización del mismo el apoyo económico para la impresión.

Finalmente, hemos contado con la generosa colaboración de los autores, quienes han brindado su valioso tiempo en la elaboración de las notas de los cursos y de los resúmenes que aquí presentamos. A todos ellos muchas gracias.

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Montevideo, Diciembre 2011.

GRUPOS ORNAMENTALES
SUBGRUPOS DISCRETOS DE LAS ISOMETRÍAS DEL PLANO

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Índice

1. Introducción

El objetivo de este cursillo es el estudio de la estructura matemática subyacente a ciertas figuras que se caracterizan por ser particularmente simétricas.

Ejemplos de este tipo de figuras se encuentran, entre otros, en la morfologí a de los seres vivos, la estructura molecular de los cristales y los diseños arquitectónicos. Esta temática ha sido un tema de interés matemático desde la antigüedad, habiéndose logrado un desarrollo importante a fines del siglo XIX. En 1900 David Hilbert presentó 23

Figura 1. Ejemplos de frisos y embaldosados.

problemas matemáticos como los más importantes de la época. Una de las tres partes del problema 18 puede ser formulada así: ¿en el espacio euclidiano n-dimensional existe solo un número finito de tipos de grupos de movimientos con región fundamental compacta? Se entiende que un tal grupo tiene asociado un subconjunto compacto K de $\mathbb{R}^n$ tal que las imágenes de K por los elementos del grupo cubren a $\mathbb{R}^n$ y sólo tienen en común puntos de sus bordes. De hecho la respuesta afirmativa para $n = 2$ y $n = 3$ ya había sido establecida alrededor de 1890 en forma independiente por Fedorov y Schoenfliess. El caso $n = 3$ tiene gran importancia práctica en el estudio de los cristales y se probó que hay 219 tipos de tales grupos. En 1910 Bieberbach fundamentó la respuesta afirmativa a la pregunta de Hilbert en toda su generalidad.

En este curso nos concentraremos en el estudio de ciertas figuras planas, que incluyen a los frisos y embaldosados periódicos (ver figura 1). El recurso principal para tales efectos es geométrico-algebraico y se basa en el estudio de ciertos subgrupos del grupo de los movimientos del plano.

La organización de estas notas es la siguiente: en la sección 2 se hace un repaso de grupos y movimientos del plano; la sección 3 contiene el estudio detallado de los grupos de Leonardo y de friso, y algunos resultados sobre los grupos de embaldosado; la sección 4 es de carácter predominantemente técnico, incluyendo algunas pruebas que preferimos separar de las secciones anteriores para agilizar la lectura. Confiamos en que el lector podrá compensar fácilmente la limitada presencia de figuras en estas notas.

2. Prerrequisitos: grupos de movimientos

Antes de empezar a trabajar con algunos subgrupos específicos del grupo de los movimientos del plano, nos conviene formular algunas precisiones sobre los grupos en general.

2.1. Grupos. Un *grupo* es un par $(G, *)$ en el cual G es un conjunto y $* : G \times G \to G$, $(g, f) \mapsto g * f$ es una función que llamamos *producto* y verifica las siguientes propiedades:

: *Asociativa*: $g * (f * h) = (g * f) * h$, para todo $g, f, h \in G$.
: *Existencia de neutro:* existe $e \in G$ tal que $g * e = e * g = g$, para todo $g \in G$.
: *Existencia de inverso:* para cada $g \in G$ existe $f \in G$ tal que $g * f = f * g = e$.

Se prueba que el neutro e inverso son únicos. Al inverso de g se lo escribe g^{-1}. En general escribiremos solo G para referirnos al grupo cuando el producto esté claro. Si G es un grupo finito, llamamos *orden* de G a su cantidad de elementos.

Un grupo G se dice *abeliano* si el producto es conmutativo, *i.e.* si verifica $g * f = f * g$, para todo $g, f \in G$.

Dos grupos G y F se dicen *isomorfos* si existe una función biyectiva $\varphi : G \to F$ que verifica $\varphi(g * g') = \varphi(g) * \varphi(g')$, para todo $g, g' \in G$. Como grupos G y F tienen las mismas propiedades y son indistinguibles.

En lo que sigue simplificaremos la notación escribiendo gf en vez de $g * f$. Un *subgrupo* de un grupo G es un subconjunto H de G que verifica:

$$e \in H, \qquad \text{si } g, f \in H \Rightarrow gf \in H, \qquad \text{si } g \in H \Rightarrow g^{-1} \in H.$$

Claramente H es un grupo con la restricción del producto de G a H.

La intersección de subgrupos de G es un subgrupo de G. El *subgrupo generado* por un subconjunto $\{g_1, \dots, g_n\}$ de G se define como la intersección de todos los subgrupos de G que lo contienen; lo denotamos $\langle g_1, \dots, g_n \rangle$. Observar que $\langle g_1, \dots, g_n \rangle$ es el menor subgrupo de G que contiene a $g_1, \dots, g_n$.
Ejemplos de grupos.

El grupo *trivial* es $G = \{e\}$, siendo $ee = e$.

Llamamos *grupo cíclico* de orden n a un grupo de la forma $G = \langle a \rangle = \{e, a, a^2, a^3, \dots, a^{n-1}\}$, en el cual se verifica $a^n = e$. Notar que G es abeliano, n es el menor natural tal que $a^n = e$ y además $a^{-1} = a^{n-1}$, $(a^2)^{-1} = a^{n-2}$, etc.

Llamamos *grupo diedral* de orden $2n$ a un grupo de la forma

$$G = \langle a, b \rangle = \left\{e, a, a^2, a^3, \dots, a^{n-1}, b, ba, ba^2, ba^3, \dots, ba^{n-1}\right\}$$

en el cual a y b verifican $a^n = b^2 = e$ y $ba^l = a^{n-l}b$, para todo $l = 0, \dots, n$. Notar que $\langle a \rangle$ es un subgrupo cíclico de G y que G no es abeliano si $n \geq 3$.

2.2. Movimientos del plano. Consideramos el plano $\mathbb{R}^2$ con su estructura de espacio vectorial con producto interno usual:

$$a(x, y) = (ax, ay), \qquad (x, y) + (x', y') = (x + x', y + y'),$$
$$\langle (x, y), (x', y') \rangle = xx' + yy', \qquad \forall a, x, x', y,' y \in \mathbb{R}.$$

Este producto interno define una norma $\| \ \|$ y esta una distancia d mediante

$$\|(x, y)\| = \sqrt{x^2 + y^2},$$
$$d\big((x, y), (x', y')\big) = \|(x, y) - (x', y')\| = \sqrt{(x - x')^2 + (y - y')^2}$$
$$\forall x, x', y,' y \in \mathbb{R}.$$

Un *movimiento* del plano es una función $M : \mathbb{R}^2 \to \mathbb{R}^2$ que preserva la distancia:

$$d\big(M(p), M(q)\big) = d(p, q), \qquad \forall p, q \in \mathbb{R}^2.$$

Llamaremos $\mathcal{M}$ al conjunto de todos los movimientos de plano. Es claro que el mapa identidad Id es un movimiento. Se prueba que todo movimiento es una función biyectiva, que el inverso de un movimiento es un movimiento y que la composición de dos movimientos es un movimiento; luego los movimientos del plano forman un grupo con la composición por producto. Si $M, N \in \mathcal{M}$, escribiremos su composición por $MN = M \circ N$. De ahora en adelante consideraremos a $\mathcal{M}$ como grupo con el producto anterior.

Es bien sabido que los movimientos del plano son las traslaciones, rotaciones, simetrías centrales, simetrías axiales y antitraslaciones. Usaremos las siguientes notaciones.

Traslación:: Si $v \in \mathbb{R}^2$, la *traslación de vector* v la escribiremos T_v. Observar que $T_o = \mathrm{Id}$.

Rotación:: Si $q \in \mathbb{R}^2$ y $\theta \in \mathbb{R}$, la *rotación de centro* q *y ángulo* θ la escribiremos $R_{q,\theta}$. Acá θ es el ángulo en radianes y la rotación es en sentido antihorario si $\theta > 0$ u horario si $\theta < 0$. Si $\theta = 0$, definimos $R_{q,0} = \mathrm{Id}$ (para todo $q \in \mathbb{R}^2$).

Simetría central:: Si $q \in \mathbb{R}^2$, la *simetría central de centro* q la escribiremos C_q. Obviamente $C_q = R_{q,\pi}$, para todo $q \in \mathbb{R}^2$, por lo cual las simetrías centrales son un caso particular de las rotaciones.

Simetría axial:: Si l es una recta en $\mathbb{R}^2$, la *simetría axial de eje* l la escribiremos S_l.

Antitraslación:: Si l es una recta y v es un vector paralelo a l en $\mathbb{R}^2$, la *antitraslación de eje* l *y vector* v la escribiremos $A_{v,l}$. Notar que $A_{v,l} = T_v S_l = S_l T_v$; en particular $A_{o,l} = S_l$, luego las simetrías axiales son un caso particular de la antitraslaciones.

Observación 2.1. Notar que la identidad es un caso particular de traslación y de rotación, así como las simetrías axiales son casos particulares de antitraslaciones. En general, cuando hablemos de una traslación o rotación, vamos a estar asumiendo que ese movimiento no es la identidad. De la misma forma si hablamos de una antitraslación estaremos asumiendo que no es una simetría axial.

Un movimiento se dice *directo* si preserva el sentido del plano e *inverso* si lo invierte. Los movimientos directos son las traslaciones, rotaciones y simetrías centrales, los inversos son las simetrías axiales y antitraslaciones. Los movimientos directos forman un subgrupo de $\mathcal{M}$.

Llamaremos $\mathcal{T}$ al conjunto de todas las traslaciones. Notar que la traslación es $T_v(p) = v + p, \ \forall p \in \mathbb{R}^2$. Vale:

$$(1) \quad T_v T_w = T_{v+w}, \qquad (T_v)^{-1} = T_{-v}, \qquad T_o = \mathrm{Id}, \qquad \forall v, w \in \mathbb{R}^2.$$

Claramente $\mathcal{T}$ es un subgrupo abeliano de $\mathcal{M}$.

Llamaremos $\mathcal{M}_q$ al conjunto de los movimientos que dejan fijo al punto q. Notar que $\mathcal{M}_q$ está formado por las rotaciones de centro en q y las simetrías axiales cuyos ejes pasan por q. Claramente $\mathcal{M}_q$ es un subgrupo de $\mathcal{M}$ (para todo $q \in \mathbb{R}^2$).

Sea $\mathcal{O} = \mathcal{M}_o$. Notar que los elementos de $\mathcal{O}$ son los movimientos que también son transformaciones lineales. Por lo anterior $\mathcal{O}$ es un subgrupo de $\mathcal{M}$ llamado el *grupo ortogonal* de $\mathbb{R}^2$.

La importancia de los subgrupos $\mathcal{T}$ y $\mathcal{O}$ de $\mathcal{M}$ es que todo elemento de $\mathcal{M}$ se escribe en forma única como producto de un elemento de $\mathcal{T}$ y uno de $\mathcal{O}$.

Teorema 2.2. *Si $M \in \mathcal{M}$, entonces existen únicos $T_v \in \mathcal{T}$ y $L \in \mathcal{O}$ tales que $M = T_v L$.*
Además $LT_v = T_{L(v)}L$, para todo $T_v \in \mathcal{T}$ y $L \in \mathcal{O}$.

Dem. Existencia: Sea $v = M(o)$ y $L = T_{-v}M$. Entonces $L(o) = (T_{-v} \circ M)(o) = -v + v = o$, luego $L \in \mathcal{M}_o = \mathcal{O}$ y $M = T_v L$.

Unicidad: supongamos que tenemos $T_v, T_w \in \mathcal{T}$ y $L, L' \in \mathcal{O}$ tales que $T_v L = T_w L'$. Entonces evaluando en o obtenemos que $v = w$, luego $T_v = T_w$ y cancelando deducimos $L = L'$.

La última afirmación se deduce de lo siguiente:

$$(LT_v)(p) = L\big(T_v(p)\big) = L(v+p) = L(v)+L(p) = T_{L(v)}\big(L(p)\big) = \big(T_{L(v)}L\big)\,(p),$$
$$\forall p \in \mathbb{R}^2. \quad \square$$

Los siguientes subgrupos de $\mathcal{M}$ son básicos para nuestro estudio.

Definición 2.3. Los grupos $\mathcal{C}_n$ y $\mathcal{D}_n$.

Sea p un punto y n un entero positivo. Si $R = R_{p,\frac{2\pi}{n}}$, entonces $R^n = \mathrm{Id}$ y

$$\mathcal{C}_n = \langle R \rangle = \left\{\mathrm{Id}, R, R^2, \ldots, R^{n-1}\right\},$$

es un subgrupo cíclico de orden n de $\mathcal{M}$. Para $n = 1, 2$ es $\mathcal{C}_1 = \{\mathrm{Id}\}$ y $\mathcal{C}_2 = \{\mathrm{Id}, C_p\}$.

Si l es una recta que pasa por p y $S = S_l$, entonces $R^n = S^2 = \mathrm{Id}$ y $SR^l = R^{n-l}S,\ \forall l = 0, \ldots, n$, luego

$$\mathcal{D}_n = \langle R, S \rangle = \left\{\mathrm{Id}, R, R^2, \ldots, R^{n-1}, S, SR, SR^2, \ldots, SR^{n-1}\right\}$$

es un subgrupo diedral de orden $2n$ de $\mathcal{M}$. Para $n = 1, 2$ es $\mathcal{D}_1 = \{\mathrm{Id}, S_l\}$ y $\mathcal{D}_2 = \{\mathrm{Id}, C_p, S_l, S_r\}$, siendo r la recta perpendicular a l por p.

Notar que $\mathcal{C}_2$ y $\mathcal{D}_1$ son isomorfos como grupos abstractos pero distintos como grupos de movimientos.

Podemos pensar que una figura es más simétrica que otra en tanto "más" movimientos la llevan sobre si misma. En este sentido un triángulo escaleno es menos simétrico que uno isósceles. La siguiente definición apunta en esa dirección.

Definición 2.4. Sea X un subconjunto de $\mathbb{R}^2$. El *grupo de simetrías de* X es el conjunto $\mathcal{M}_X$ formado por los movimientos del plano que dejan fijo al conjunto X, es decir

$$\mathcal{M}_X = \{M \in \mathcal{M} :\ M(X) = X\}.$$

Claramente $\mathcal{M}_X$ es un subgrupo de $\mathcal{M}$. Notar que $\mathcal{M}_q = \mathcal{M}_{\{q\}}$, para todo $q \in \mathbb{R}^2$.

Ejemplo 2.5. Sea $\mathcal{P}_n$ un polígono regular de n lados. Si p es el centro de $\mathcal{P}_n$ y l es una recta que pasa por p y por alguno de los vértices de $\mathcal{P}_n$, entonces es fácil de probar que el grupo de simetrías de $\mathcal{P}_n$ es $\mathcal{D}_n = \langle R, S \rangle$, siendo $R = R_{p,\frac{2\pi}{n}}$ y $S = S_l$. Por otro lado el grupo de las simetrías "directas" de $\mathcal{P}_n$ (es decir, los elementos de $\mathcal{M}_{\mathcal{P}_n}$ que además son movimiento directos) es $\mathcal{C}_n = \langle R \rangle$.

3. Grupos ornamentales

Ahora vamos a definir los subgrupos de $\mathcal{M}$ con los cuales se construyen en el plano los diseños que nos interesan.

Definición 3.1. Un *grupo ornamental* es un subgrupo G de $\mathcal{M}$ que verifica las siguientes condiciones:

GO1:: Existe $k > 0$ tal que si $T_v \in G$ y $v \neq 0$, entonces $\|v\| \geq k$.
GO2:: Existe $h > 0$ tal que si $R_{p,\theta} \in G$ con $0 \leq \theta < 2\pi$, entonces $\theta \geq h$.

Observación 3.2. Si G es un grupo ornamental, entonces G no puede tener traslaciones de vectores arbitrariamente pequeños ni rotaciones con ángulos arbitrariamente pequeños. Como el producto de dos simetrías axiales es una traslación o una rotación, entonces tampoco puede tener simetrías axiales de ejes paralelos muy próximos o que se corten formando ángulos muy chicos. Como el cuadrado de una antitraslación es una traslación, tampoco puede tener antitraslaciones de vectores arbitrariamente pequeños.

Ejercicio 3.3. Fijada una base ortonormal de $\mathbb{R}^2$ es sabido que se tiene un isomorfismo entre los movimientos que son transformaciones lineales y las matrices ortogonales reales 2x2. Esto junto con el teorema 2.2 permite definir una distancia entre los elementos de $\mathcal{M}$ pensados como elementos de $\mathbb{R}^6$ donde usamos la distancia habitual. Se dice que un subgrupo G de $\mathcal{M}$ es *discreto* si es discreto como subconjunto de $\mathbb{R}^6$. Probar que si G es un subgrupo discreto de $\mathcal{M}$, entonces se verifican **GO1** y **GO2**.

3.1. Grupos de Leonardo.

Definición 3.4. Un grupo ornamental que no contiene traslaciones se llama un *grupo de Leonardo*[1].

Observación 3.5. Si G es $\mathcal{C}_n$ o $\mathcal{D}_n$, entonces G no contiene traslaciones y el ángulo mínimo de una rotación que esté en G es $\frac{2\pi}{n}$. Luego $\mathcal{C}_n$ y $\mathcal{D}_n$ son grupos de Leonardo. El siguiente teorema muestra que no hay otros.

Teorema 3.6. *Si G es un grupo de Leonardo, entonces es el grupo cíclico $\mathcal{C}_n$ o el diedral $\mathcal{D}_n$, para algún n.*

Dem. Empezamos estudiando el tipo de movimientos que pueden haber en G.

- Como el cuadrado de una antitraslación es una traslación, deducimos que G no contiene antitraslaciones.
- En la proposición 4.3 vimos que si R_1 y R_2 son dos rotaciones con centros distintos, entonces $R_1 R_2 R_1^{-1} R_2^{-1}$ es una traslación. Luego si G tiene rotaciones, estas tienen que tener el mismo centro.
- El producto de dos simetrías axiales de ejes paralelos es una traslación, luego si G tiene simetrías axiales, los ejes de estas necesariamente se cortan. Observar que si los ejes de tres simetrías forman un triángulo, entonces el producto de las tres es una antitraslación, luego los ejes de las simetrías axiales de G (en caso de existir) pasan todos por un mismo punto.
- La composición de una simetría axial con una rotación de centro que no está en el eje de la simetría es una antitraslación, luego si en G hay rotaciones y simetrías axiales, entonces todas las rotaciones tienen un mismo centro p y todas los ejes de las simetrías pasan por p.

Si G está formado solo por la identidad, es $\mathcal{C}_1$ y si G está formado por la identidad y una simetría axial, es $\mathcal{D}_1$. Si G no es de las formas anteriores, entonces necesariamente contiene una rotación. En lo que sigue nos situamos en este caso.

[1]El nombre viene de Leonardo da Vinci.

Sea R_G el conjunto de las rotaciones que están en G y p el centro común de las mismas. Vale

$$R_{p,\theta_1}R_{p,\theta_2} = R_{p,\theta_1+\theta_2}, \qquad (R_{p,\theta})^{-1} = R_{p,-\theta}, \qquad R_{p,0} = \mathrm{Id},$$
$$\forall \theta, \theta_1, \theta_2 \in \mathbb{R}.$$

Luego R_G es un subgrupo de G. Esto implica que $H = \{\theta \in \mathbb{R} : R_{p,\theta} \in R_G\}$ es un subgrupo de $\mathbb{R}$ y por lo tanto aplicando el teorema 4.1 deducimos que existe $a \in \mathbb{R}$ tal que $H = \mathbb{Z}a$ o H es denso en $\mathbb{R}$. En el segundo caso tendríamos que 0 es un punto de acumulación de H contradiciendo **GO2**, luego existe $a \in \mathbb{R}$ tal que $H = \mathbb{Z}a$. Sabemos que R_G no es trivial, así que $a \neq 0$ y podemos asumir $a > 0$.

Observar que $2\pi \in H$, luego existe un entero n (necesariamente positivo) tal que $2\pi = na$ y por lo tanto $a = \frac{2\pi}{n}$. Sea $R = R_{p,\frac{2\pi}{n}} \in R_G$, luego $\mathcal{C}_n = \langle R \rangle = \{\mathrm{Id}, R, R^2, \ldots, R^{n-1}\} \subset R_G$. Probaremos $R_G = \mathcal{C}_n$.

Si $R_{p,\theta} \in R_G$, entonces $\theta \in H = \mathbb{Z}\frac{2\pi}{n}$. Luego existe $m \in \mathbb{Z}$ tal que $\theta = \frac{2\pi m}{n}$ y por lo tanto

$$R_{p,\theta} = R_{p,\frac{2\pi m}{n}} = \left(R_{p,\frac{2\pi}{n}}\right)^m = R^m \in \langle R \rangle.$$

Esto concluye la prueba de $R_G = \mathcal{C}_n$.

Si G no contiene simetrías axiales, entonces $G = R_G = \mathcal{C}_n$. Supongamos ahora que G contiene una simetría axial S. Como $R, S \in G$, es $\mathcal{D}_n = \langle R, S \rangle \subset G$. Probaremos $G = \mathcal{D}_n$.

Sea $M \in G$. Si M es una rotación, entonces $M \in R_G = \langle R \rangle \subset \langle R, S \rangle$. Si M es una simetría axial, entonces $MS \in G$ es una rotación y por lo tanto existe m tal que $MS = R^m$; luego $M = R^m S^{-1} = R^m S \in \langle R, S \rangle$. Esto completa la prueba de $G = \langle R, S \rangle$. □

Un dibujo cuyo grupo de simetrías es un grupo de Leonardo, es el de la figura 2.

Corolario 3.7. *Si G es un subgrupo finito de $\mathcal{M}$, entonces G es el grupo cíclico $\mathcal{C}_n$ o el diedral $\mathcal{D}_n$, para algún n.*

FIGURA 2. Grupo de Leonardo.

Dem. Como G es finito, no puede contener traslaciones ($(T_v)^n = T_{nv}$, $\forall n \in \mathbb{Z}$) y como la cantidad de rotaciones que hay en G es finita, entonces es claro que verifica **GO2**. Luego G es un grupo de Leonardo y se aplica el teorema anterior. $\square$

Observación 3.8. El corolario anterior sugiere la siguiente disgresión: ¿cuáles son los subgrupos finitos de los movimientos del espacio tridimensional? Un teorema debido a Félix Klein asegura que los formados por movimientos directos son los grupos $\mathcal{C}_n$, $\mathcal{D}_n$ y los grupos de simetría de los poliedros regulares.

3.2. Grupos asociados a un grupo ornamental. De ahora en adelante G es un grupo ornamental que contiene traslaciones. En esta sección introduciremos dos subgrupos que denotaremos $\mathcal{T}_G$ y $\tilde{G}$ asociados a cada grupo ornamental G.

Sea $\mathcal{T}_G$ el grupo de las traslaciones de G, es decir $\mathcal{T}_G = \mathcal{T} \cap G$. La función $\phi : \mathbb{R}^2 \to \mathcal{T}$ definida por $\phi(v) = T_v$ para todo $v \in \mathbb{R}^2$, es un isomorfismo de grupos considerando $\mathbb{R}^2$ como grupo con la suma (recordar las fórmulas (1)). Via este isomorfismo a $\mathcal{T}_G$ le corresponde el subgrupo L_G de $\mathbb{R}^2$ definido por

$$L_G = \left\{ v \in \mathbb{R}^2 : \ T_v \in G \right\}.$$

Observar que estamos asumiendo $\mathcal{T}_G \neq \{\mathrm{Id}\}$, luego $L_G \neq \{0\}$.

Lema 3.9. *Si k es como en* **GO1***, entonces para todo $v, w \in L_G$ con $v \neq w$, es $d(v, w) \geq k$.*

Dem. Si existiesen $v, w \in L_G$ tal que $d(v, w) < k$ y $w \neq v$, entonces definiendo $u = v - w$, es $u \in L_G$, $\|u\| < k$ y $u \neq 0$. Como $u \in L_G$, entonces $T_u \in G$ pero esto contradice **GO1** ya que $\|u\| < k$ y $u \neq 0$. □

Lema 3.10. *Si $\mathcal{P} \subset \mathbb{R}^2$ es un conjunto acotado, entonces $\mathcal{P} \cap L_G$ es finito.*

Dem. Si $\mathcal{P} \cap L_G$ fuese infinito, al ser acotado existiría un punto de acumulación p de $\mathcal{P} \cap L_G$. Luego si k es como en **GO1**, podemos encontrar dos elementos distintos v, w en $\mathcal{P} \cap L_G$ que distan de p menos que $\frac{k}{2}$; luego v, w están en L_G y distan entre sí menos de k contradiciendo el lema 3.9. □

Lema 3.11. *Si $v \in L_G$ y $v \neq 0$, entonces existe $0 \neq w \in L_G$, tal que $\mathbb{R}v \cap L_G = \mathbb{Z}w$.*

Dem. Sea $H = \{a \in \mathbb{R} : \; av \in L_G\}$. Como L_G es un subgrupo de $\mathbb{R}^2$, entonces H es un subgrupo de $\mathbb{R}$. Luego aplicando el teorema 4.1 sabemos que existe $a_0 \in \mathbb{R}$ tal que $H = \mathbb{Z}a_0$ o H es denso en $\mathbb{R}$. Si fuese el segundo caso, y k es como en **GO1**, existiría $a \in H$ tal que $1 < a < 1 + \frac{k}{\|v\|}$. Pero esto implica $T_{av-v} \in G$, $\|av - v\| < k$ y $av - v \neq 0$ contradiciendo **GO1**.

Luego existe $a_0 \in \mathbb{R}$ tal que $\{a \in \mathbb{R} : \; av \in L_G\} = \mathbb{Z}a_0$ y esto implica $\mathbb{R}v \cap L_G = \mathbb{Z}w$, siendo $w = a_0 v$. Al ser $0 \neq v \in \mathbb{R}v \cap L_G = \mathbb{Z}w$, se deduce $w \neq 0$. □

Teorema 3.12. *Si G es un grupo ornamental que contiene traslaciones, entonces L_G solo puede tener una de las formas siguientes:*

1. *$\mathbb{Z}v = \{nv : \; n \in \mathbb{Z}\}$, $v \neq 0$.*
2. *$\mathbb{Z}v + \mathbb{Z}w = \{nv + mw : \; n, m \in \mathbb{Z}\}$, con v, w linealmente independientes.*

Dem. Sea $v_0 \in L_G$, $v_0 \neq 0$. Aplicando el lema 3.11 tenemos que existe $0 \neq v \in L_G$ tal que $\mathbb{R}v_0 \cap L_G = \mathbb{Z}v$. Luego $\mathbb{Z}v$ es un subgrupo de L_G. Si $\mathbb{Z}v = L_G$, entonces estamos en el primer caso.

Supongamos ahora que $\mathbb{Z}v \subsetneq L_G$. Luego existe $w_0 \in L_G$ tal que $w_0 \notin \mathbb{Z}v$. Entonces

$$w_0 \notin \mathbb{R}v_0 \cap L_G = \mathbb{R}v \cap L_G \quad \Rightarrow \quad w_0 \notin \mathbb{R}v$$
$$\Rightarrow \quad \{v, w_0\} \text{ es linealmente independiente.}$$

Sea $P_0 = \{av + bw_0 : \ a, b \in [0,1]\}$ el paralelogramo generado por v y w_0. Si $u \in P_0$, es $\|u\| \leq \|v\| + \|w_0\|$, luego P_0 está acotado y por lo tanto $Q_0 = \{u \in P_0 : \ u \notin \mathbb{R}v\}$ está acotado. Luego el lema 3.10 implica que $Q_0 \cap L_G$ es finito. Sea $w \in Q_0 \cap L_G$ tal que

$$d(w, \mathbb{R}v) = \text{mín}\{d(u, \mathbb{R}v) : \ u \in Q_0 \cap L_G\}.$$

Notar que v, w son linealmente independientes. Sean $P = \{av + bw : a, b \in [0,1]\}$ y $Q = \{u \in P : \ u \notin \mathbb{R}v\}$.

Afirmación 1: si $z \in Q \cap L_G$, entonces $d(z, \mathbb{R}v) \geq d(w, \mathbb{R}v)$.

Supongamos que existe $z \in Q \cap L_G$ tal que $d(z, \mathbb{R}v) < d(w, \mathbb{R}v)$. Como $z \in P$ y $w \in P_0$, entonces existen $a_0, b_0, a_1, b_1 \in [0,1]$ tales que

$$w = a_0 v + b_0 w_0, \quad z = a_1 v + b_1 w, \quad \Rightarrow \quad z = (a_1 + a_0 b_1)v + (b_0 b_1)w_0.$$

Como $z \in L_G$ y $z \notin \mathbb{R}v$, entonces necesariamente $z \notin P_0$. Pero $b_0 b_1 \in [0,1]$ y $a_1 + a_0 b_1 \in [0,2]$, luego es $1 < a_1 + a_0 b_1 \leq 2$. Entonces

$$z - v = (a_1 + a_0 b_1 - 1)v + (b_0 b_1)w_0 \in L_G \quad \text{y} \quad 0 < a_1 + a_0 b_1 - 1 \leq 1$$
$$\Rightarrow \quad z - v \in Q_0 \cap L_G.$$

Pero la recta que une z con $z-v$ es paralela a $\mathbb{R}v$, luego $d(z-v, \mathbb{R}v) = d(z, \mathbb{R}v) < d(w, \mathbb{R}v)$ contradiciendo la elección de w.

Afirmación 2: $P \cap L_G = \{o, v, w, v + w\}$.

Observar que si $z \in P$, claramente es $d(z, \mathbb{R}v) \leq d(w, \mathbb{R}v)$; luego por la afirmación 1 es $d(z, \mathbb{R}v) = d(w, \mathbb{R}v)$, para todo $z \in Q \cap L_G$. Entonces $Q \cap L_G \subset \{w + cv : \ c \in [0,1]\}$.

Si $z \in Q \cap L_G$, entonces existe $c \in [0,1]$ tal que $z = w + cv$, luego

$$cv = z - w \in L_G \quad \Rightarrow \quad cv \in L_G \cap \mathbb{R}v = \mathbb{Z}v \quad \Rightarrow \quad c = 0, 1$$
$$\Rightarrow \quad Q \cap L_G = \{w, \ w + v\}.$$

Por otro lado, $P \cap L_G \cap \mathbb{R}v = P \cap \mathbb{Z}v = \{o, v\}$. Luego

$P \cap L_G = \big(Q \cup (P \cap \mathbb{R}v)\big) \cap L_G = (Q \cap L_G) \cup (P \cap \mathbb{R}v \cap L_G) = \{o, v, w, v+w\}$.

Afirmación 3: $L_G = \mathbb{Z}v + \mathbb{Z}w$.

Claramente $\mathbb{Z}v + \mathbb{Z}w \subset L_G$. Sea $z \in L_G$. Al ser $\{v, w\}$ un subconjunto linealmente independiente de $\mathbb{R}^2$ es una base, luego existen $a, b \in \mathbb{R}$ tales que $z = av + bw$. Tomando la parte entera de a y b tenemos que existen $m, n \in \mathbb{Z}$ y $r, s \in [0, 1)$ tales que $a = m + r$ y $b = n + s$. Luego

$$z - (mv + nw) = (a - m)v + (b - n)w = rv + sw$$
$$\Rightarrow \quad rv + sw \in P \cap L_G = \{o, v, w, v + w\}$$
$$\Rightarrow r = s = 0 \quad \Rightarrow \quad z = mv + nw \in \mathbb{Z}v + \mathbb{Z}w. \quad \square$$

El teorema anterior implica que si un grupo ornamental G contiene traslaciones, entonces $\mathcal{T}_G$ es de la forma $\langle T_v \rangle$ o $\langle T_v, T_w \rangle$, con v y w linealmente independientes. Esto da lugar a la siguiente definición.

Definición 3.13.

1. Un grupo ornamental G se llama un *grupo de friso* si existe $v \neq 0$ tal que
$$\mathcal{T}_G = \langle T_v \rangle = \{T_{nv} : n \in \mathbb{Z}\}.$$
2. Un grupo ornamental G se llama un *grupo de embaldosado*[2] si existen v, w linealmente independientes tales que
$$\mathcal{T}_G = \langle T_v, T_w \rangle = \{T_{nv+mw} : n, m \in \mathbb{Z}\}.$$

Observación 3.14. De acuerdo al teorema anterior los grupos ornamentales se clasifican en grupos de Leonardo, de friso o de embaldosado. En el teorema 3.6 vimos cómo era la forma de los grupos de Leonardo, en lo que sigue estudiaremos la clasificación de los grupos de friso y daremos un pantallazo sobre los grupos de embaldosado, cuyo estudio excede los objetivos del curso.

[2]A los grupos de embaldosado también se les llama *grupos de empapelado.*

Recordar que en el teorema 2.2 probamos que si $M \in \mathcal{M}$, entonces existen únicos $T_v \in \mathcal{T}$ y $L \in \mathcal{O}$ tales que $M = T_v L$. Si G es un grupo ornamental, definimos

$$\tilde{G} = \{L \in \mathcal{O} : \ \exists\, T_v \in \mathcal{T} \text{ tal que } T_v L \in G\}.$$

Proposición 3.15. *Si G es un grupo ornamental, entonces $\tilde{G}$ es un grupo de Leonardo.*

Dem. Es claro que la identidad está en $\tilde{G}$ ($\mathrm{Id} = T_o\,\mathrm{Id}$). Si $L \in \tilde{G}$, entonces existe T_v en $\mathcal{T}$ tal que $T_v L \in G$. Luego aplicando la proposición 2.2 obtenemos

$$G \ni (T_v L)^{-1} = L^{-1} T_v^{-1} = L^{-1} T_{-v} = T_{-L^{-1}(v)} L^{-1} \qquad \Rightarrow \qquad L^{-1} \in \tilde{G}.$$

Si $L, L' \in \tilde{G}$, entonces existen T_v, T_w en $\mathcal{T}$ tal que $T_v L, T_w L' \in G$. Luego

$$G \ni (T_v L)\,(T_w L') = T_v T_{L(w)} L L' = T_{v+L(w)} L L' \qquad \Rightarrow \qquad L L' \in \tilde{G}.$$

Esto prueba que $\tilde{G}$ es un subgrupo de $\mathcal{M}$. Como en $\tilde{G}$ no hay traslaciones, para ver que es un grupo de Leonardo solo hay que probar que verifica **GO2**. Si $R_{o,\theta} \in \tilde{G}$, entonces existe $T_v \in \mathcal{T}$ tal que $T_v R_{o,\theta} \in G$. Pero $T_v R_{o,\theta} = R_{p,\theta}$ para algún p, y como G verifica **GO2** deducimos que $\tilde{G}$ también. $\square$

Si G es un grupo ornamental, definimos $\tilde{G} \cdot L_G = \{L(v) : \ L \in \tilde{G},\ v \in L_G\}$.

Teorema 3.16. *Si G es un grupo ornamental, entonces $\tilde{G}$ deja invariante a L_G, es decir $\tilde{G} \cdot L_G = L_G$.*

Dem. Es claro que $L_G \subset \tilde{G} \cdot L_G$, porque $\mathrm{Id} \in \tilde{G}$. Para probar la inclusión en sentido contrario, sea $u \in \tilde{G} \cdot L_G$, entonces existe $L \in \tilde{G}$ y $v \in L_G$ tales que $u = L(v)$. Luego $T_v \in G$ y existe $T_w \in \mathcal{T}$ tal que $T_w L \in G$. Si llamamos $M = T_w L \in G$, es $L = T_w^{-1} M$.

Queremos probar que $u \in L_G$, es decir que $T_u \in G$. Observar que aplicando la proposición 2.2 tenemos $L T_v = T_{L(v)} L = T_u L$, luego

$$T_u = L T_v L^{-1} = \left(T_w^{-1} M\right) L_v \left(T_w^{-1} M\right)^{-1} = T_w^{-1} M T_v M^{-1} T_w$$

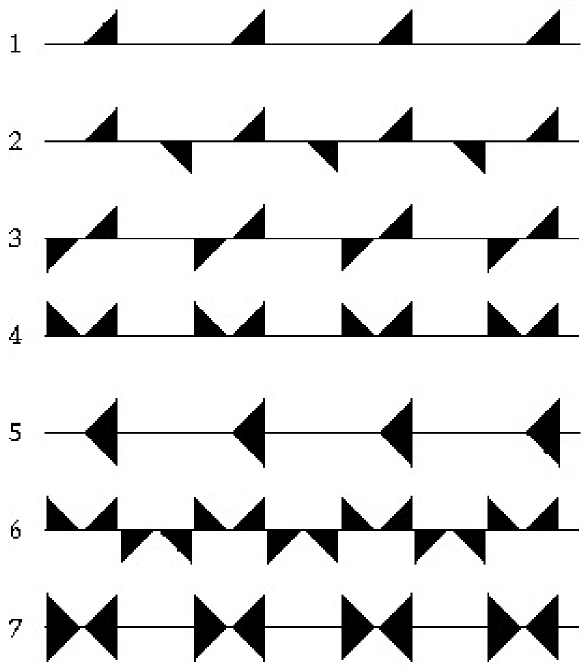


Figura 3. Los 7 grupos de friso.

$$\Rightarrow \quad MT_vM^{-1} = T_wT_uT_w^{-1} = T_{w+u-w} = T_u.$$

Como G es un grupo y $M, T_v \in G$, deducimos que $T_u = MT_vM^{-1} \in G$. Luego $u \in L_G$. □

3.3. Grupos de friso. Ahora con los resultados que tenemos estamos en condiciones de describir explícitamente los grupos de friso. Primero para un grupo de friso G, usando el teorema 3.16 determinamos todos los $\tilde{G}$ posibles; luego el teorema 4.2 limita las posibilidades para los G que generan a tales $\tilde{G}$; finalmente sólo resta confirmar cuáles de estos G encontrados son efectivamente grupos de friso. La conclusión es que hay exactamente siete grupos de friso, estos pueden

FIGURA 4. Grupos de friso.

ser representados de manera sencilla como se muestra en la figura 3. En cada caso, el grupo de friso es el grupo de simetrías de la figura.

La prueba del siguiente teorema se encuentra en la Sección 4.

Teorema 3.17. *Si G es un grupo de friso con $\mathcal{T}_G = \langle T_v \rangle$, entonces G es uno de los siguientes:*

- $\langle T_v \rangle = \{T_{nv} : \ n \in \mathbb{Z}\}$,
- $\langle T_v, C_p \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{C_{p+\frac{n}{2}v} : \ n \in \mathbb{Z}\right\}$,
- $\langle T_v, S_l \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \{A_{nv,l} : \ n \in \mathbb{Z}\} \cup \{S_l\}, \quad l \parallel v$,
- $\left\langle A_{\frac{v}{2},l} \right\rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{A_{\left(n+\frac{1}{2}\right)v,l} : \ n \in \mathbb{Z}\right\}$,
- $\langle T_v, S_r \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \{S_{r_n} : \ n \in \mathbb{Z}\}, \quad r \perp v, \ r_n = T_{\frac{n}{2}v}(r), \ \forall n$,
- $\langle T_v, S_r, S_l \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{C_{p+\frac{n}{2}v} : \ n \in \mathbb{Z}\right\} \cup \{S_{r_n} : \ n \in \mathbb{Z}\} \cup$
  $\cup \{A_{nv,l} : \ n \in \mathbb{Z}\} \cup \{S_l\}$,
  $l \parallel v, \ r \perp v, \ p = l \cap r, \ r_n = T_{\frac{n}{2}v}(r), \ \forall n$,
- $\left\langle S_r, C_q, A_{\frac{v}{2},l} \right\rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{C_{q+\frac{n}{2}v} : \ n \in \mathbb{Z}\right\} \cup \{S_{r_n} : \ n \in \mathbb{Z}\} \cup$
  $\cup \left\{A_{\left(n+\frac{1}{2}\right)v,l} : \ n \in \mathbb{Z}\right\}$,
  $l \parallel v, \ r \perp v, \ p = l \cap r, \ q = T_{\frac{v}{4}}(p), \ r_n = T_{\frac{n}{2}v}(r), \ \forall n. \quad \square$

Ejercicio 3.18. Identificar los grupos de friso en las figuras 3 y 4.

3.4. Conjuntos fundamentales.

Definición 3.19. Si G es un subgrupo de los movimientos del plano, las *órbitas* de G son los conjuntos de la forma $G \cdot x = \{M(x) : \ M \in G\}$, con $x \in \mathbb{R}^2$.

Ejercicio 3.20. Probar que las órbitas de G constituyen una partición del plano, es decir, la unión de las órbitas es todo el plano y dos órbitas o son disjuntas o coinciden.

Definición 3.21. Si G es un subgrupo de los movimientos del plano, un *conjunto fundamental* para G es un conjunto F de puntos del plano que contiene exactamente un punto de cada órbita de G.

El interés en los conjuntos fundamentales, es que si tenemos un dibujo en un conjunto fundamental para un grupo ornamental, transformando este conjunto fundamental con los movimientos del grupo se lograría llevar el dibujo a todo el plano sin que ocurran solapamientos.

Dado un grupo ornamental G, pretendemos obtener un método que nos permita conseguir algún conjunto fundamental para G que tenga una forma sencilla.

Un procedimiento para encontrar un conjunto fundamental de este tipo es el siguiente: primero fijamos un punto p del plano que no quede fijo por ninguno de los elementos de $G^* := G \setminus \{\mathrm{Id}\}$. Luego, se puede probar que en cada órbita existe al menos un punto a una distancia mínima de p. Observar que un punto x de una órbita cumple la condición anterior si $d(x,p) \leq d(M(x),p)$, $\forall M \in G$; dado que G está formado por movimientos, esto es equivalente a que $d(x,p) \leq d\big(x, M(p)\big)$, $\forall M \in G$. Notar que si $M(p) \neq p$, entonces cada una de estas desigualdades define un semiplano.

Seamos más precisos. Sea G un grupo ornamental y elegimos un punto p del plano que no quede fijo por ninguno de los elementos de G^*. Consideremos el conjunto

$$D = \left\{x \in \mathbb{R}^2 : \ d(x,p) < d\big(x, M(p)\big), \ \forall M \in G^*\right\}.$$

Observar que D es convexo por ser intersección de semiplanos abiertos y es no es vacío puesto que p está en todos estos semiplanos, dado que $M(p) \neq p$, $\forall M \in G^*$.

Proposición 3.22. *El conjunto D contiene a lo sumo un punto de cada órbita de G.*

Dem. Supongamos que tenemos dos puntos de una misma órbita en D, es decir x y $M(x)$ pertenecen a D con $M \in G$, notar que $M \neq \mathrm{Id}$. Usando que M es un movimiento obtenemos que

$$d\left(x, M^{-1}(p)\right) = d\big(M(x), p\big) < d\big(M(x), S(p)\big) = d\left(x, M^{-1}\big(S(p)\big)\right),$$

para todo $S \in G^*$. Tomando $M = S$ obtenemos $d\left(x, M^{-1}(p)\right) < d(x,p)$ lo cual es absurdo puesto que $d(x,p) < d\left(x, S(p)\right)$, para todo $S \in G^*$. □

El inconveniente del resultado anterior es que pueden existir órbitas que no corten a D. Esa deficiencia la repara el siguiente teorema que presentamos sin demostración.

Teorema 3.23. *Existe un conjunto fundamental F que contiene a D y está contenido en el conjunto*

$$R = \left\{x \in \mathbb{R}^2 :\ d(x,p) \leq d\big(x, M(p)\big),\ \forall M \in G^*\right\}.$$

□

Ejercicio 3.24. Hallar un conjunto fundamental para cada grupo de friso.

3.5. Grupos de embaldosado. Para grupos de embaldosado describiremos su parte lineal, en el sentido del teorema 2.2.

Proposición 3.25. *Si G un grupo de embaldosado, entonces $\tilde{G}$ solo puede ser el grupo cíclico $\mathcal{C}_n$ o el diedral $\mathcal{D}_n$, para algún $n = 1, 2, 3, 4, 6$.*

Dem. Sabemos por el teorema 3.15 que $\tilde{G}$ es un grupo de Leonardo y por lo tanto es $\mathcal{C}_n$ o $\mathcal{D}_n$, para algún $n = 1, 2, \ldots$. Supongamos que $n \geq 3$. Sea v un vector no nulo de norma mínima en L_G y $R = R_{o,\theta}$, $\theta = \frac{2\pi}{n}$, el generador del grupo de rotaciones de $\tilde{G}$. La condición $n \geq 3$ implica $0 < \theta < \pi$. Como $\tilde{G}$ deja L_G invariante, es $R(v) \in L_G$ y por lo tanto $R(v) - v \in L_G$. Considerando el triángulo isósceles de vértices o, v y $R(v)$, deducimos

$$\operatorname{sen}\left(\frac{\theta}{2}\right) = \frac{\|R(v) - v\|}{2\|v\|} \geq \frac{\|v\|}{2\|v\|} = \frac{1}{2}, \quad 0 < \frac{\theta}{2} < \frac{\pi}{2} \quad \Rightarrow \quad \frac{\theta}{2} \geq \frac{\pi}{6}$$

$$\Rightarrow \quad \frac{2\pi}{n} = \theta \geq \frac{\pi}{3} \quad \Rightarrow \quad n \leq 6.$$

Por otro lado, el punto medio entre v y $R^2(v)$ es $\frac{1}{2}\big(v + R^2(v)\big)$. Observar que $v + R^2(v) \in L_G$, luego si consideramos el triángulo isósceles de vértices o, v y $R^2(v)$, deducimos

$$\cos(\theta) = \frac{\|R^2(v) + v\|}{2\|v\|} \geq \frac{\|v\|}{2\|v\|} = \frac{1}{2} \quad \Rightarrow \quad \cos\left(\frac{2\pi}{n}\right) \geq \frac{1}{2}.$$

Calculando $\cos\left(\frac{2\pi}{n}\right)$ para $n = 3, 4, 5, 6$ obtenemos

$$\cos\left(\frac{2\pi}{3}\right) = 0,5 \quad \cos\left(\frac{\pi}{2}\right) \cong 0,7 \quad \cos\left(\frac{2\pi}{5}\right) \cong 0,3 \quad \cos\left(\frac{\pi}{3}\right) \cong 0,9.$$

Esto implica que no puede ser $n = 5$, luego los únicos valores posibles de n son 1, 2, 3, 4 y 6. □

La figura 5 muestra dos ejemplos de dibujos cuyos grupos de simetrías son grupos de embaldosado.

El siguiente es una versión del teorema 3.17 para grupos de embaldosado, que muestra que las 10 posibles opciones de $\tilde{G}$ (dadas por la proposición anterior) dan lugar a 17 grupos de embaldosado. Como ya comentamos en la introducción, este resultado fue obtenido alrededor de 1890 por Fedorov y Schoenfliess.

Teorema 3.26. *La siguiente es una lista de generadores para cada grupo de embaldosado.*

1. *Dos traslaciones de vectores linealmente independientes.*
2. *Tres simetrías centrales con centros no alineados.*
3. *Dos simetrías axiales y una traslación de ejes paralelos.*
4. *Dos antitraslaciones de ejes paralelos.*
5. *Una simetría axial y una antitraslación de ejes paralelos.*
6. *Simetrías axiales respecto a los cuatro lados de un rectángulo.*
7. *Una simetría axial y dos simetrías centrales con centros equidistantes del eje.*
8. *Dos antitraslaciones de ejes perpendiculares.*
9. *Dos simetrías axiales de ejes perpendiculares y una simetría central.*

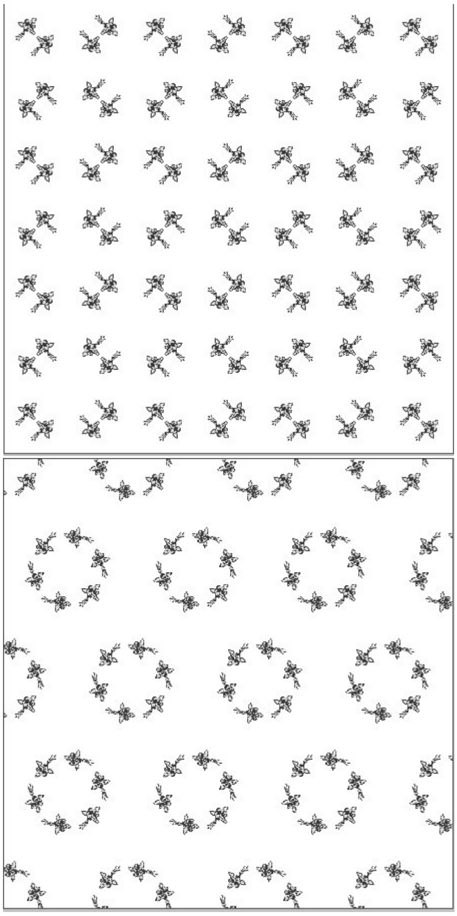


FIGURA 5. Grupos de embaldosado.

10. *Una simetría central y una rotación de ángulo* $\frac{\pi}{2}$.
11. *Simetrías axiales respecto a los tres lados de un triángulo de ángulos* $\frac{\pi}{4}$, $\frac{\pi}{4}$ *y* $\frac{\pi}{2}$.
12. *Una simetría axial y una rotación de ángulo* $\frac{\pi}{2}$.
13. *Dos rotaciones de ángulo* $\frac{2\pi}{3}$.
14. *Una simetría axial y una rotación de ángulo* $\frac{2\pi}{3}$.
15. *Simetrías axiales respecto a los tres lados de un triángulo equilátero.*

16. *Una simetría central y una rotación de ángulo* $\frac{2\pi}{3}$.
17. *Simetrías axiales respecto a los tres lados de un triángulo de ángulos* $\frac{\pi}{6}$, $\frac{\pi}{3}$ *y* $\frac{\pi}{2}$. $\square$

4. Anexo

El siguiente teorema clasifica los subgrupos aditivos del cuerpo de números reales.

Teorema 4.1. *Si G es un subgrupo de $(\mathbb{R}, +)$, entonces existe $\alpha \in \mathbb{R}$ tal que $G = \mathbb{Z}\alpha = \{n\alpha : \ n \in \mathbb{Z}\}$ o G es denso en $\mathbb{R}$.*

Dem. Si $G = \{0\}$ el resultado es válido con $\alpha = 0$. De aquí en más suponemos que este no es el caso.

Sea $\alpha = \text{ínf}\{|x| : \ x \in G, \ x \neq 0\}$. Una primera posibilidad es que $\alpha = \text{mín}\{|x| : \ x \in G, \ x \neq 0\}$, aquí $\alpha > 0$. Dado $x \in G$ escribimos $x = \alpha n + y$, donde $n \in \mathbb{Z}$ con $0 \leq y < \alpha$; esto implica que $y = x - \alpha n \in G$, lo cual contradice la definición de α, a menos que $y = 0$. Luego $G = \mathbb{Z}\alpha$.

La segunda y última posibilidad es que $\alpha \notin \{|x| : \ x \in G, \ x \neq 0\}$. En este caso vamos a probar que G es denso en $\mathbb{R}$, es decir que si (a, b) es un intervalo de $\mathbb{R}$ se tiene que $G \cap (a, b) \neq \phi$. Sea $\epsilon = b - a > 0$. Necesariamente existen $y, z \in G$ tales que $\alpha < y < z < \alpha + \epsilon$, luego $x := z - y \in G$ y $0 < x < \epsilon$. Esto implica que existe $N \in \mathbb{Z}$ tal que $Nx \in (a, a + \epsilon) = (a, b)$. $\square$

Para aplicar el teorema 2.2 es útil la siguiente proposición. Su demostración así como la de la siguiente quedan como ejercicio.

Proposición 4.2. *Sea $T_v \in \mathcal{T}$. Consideremos $R_{o,\theta}$, C_o, S_l (l recta que pasa por o) elementos genéricos de $\mathcal{O}$. Entonces existen un vector w, un puntos p y rectas r, s paralelas a l, tales que*

$$T_v R_{o,\theta} = R_{p,\theta}, \qquad T_v C_o = C_p,$$

$$T_v S_l = \begin{cases} S_r, & \text{si } v \perp l, \text{ con } r = T_{\frac{v}{2}}(l) \\ A_{w,s} & \text{si } v \not\perp l, \text{ con } s = T_{\frac{v}{2}}(l) \end{cases}.$$

Luego la correspondencia $\mathcal{O} \to G$ dada por $L \mapsto T_v L$ lleva una rotación en otra rotación (con el mismo ángulo de giro), la simetría *central en una simetría central (con otro centro) y una simetría axial en otra simetría axial o en una antitraslación (en ambos casos con ejes paralelos al de la simetría original).* □

La siguiente proposición resume algunas propiedades de la composición de movimientos (observar que la proposición anterior es un caso particular de esta).

Proposición 4.3.

1. *La composición de una traslación con una rotación es una rotación con otro centro y el mismo ángulo:*
$$T_v R_{p,\theta} = R_{q,\theta},$$
siendo $q = T_w(p)$. El vector w se obtiene como la hipotenusa del triángulo que tiene al vector $\frac{v}{2}$ por cateto y $\frac{\theta}{2}$ como ángulo opuesto a $\frac{v}{2}$.
2. *La composición de una traslación con una simetría central es una simetría central con otro centro:*
$$C_p T_v = C_{p-\frac{v}{2}}, \qquad T_v C_p = C_{p+\frac{v}{2}}.$$
3. *La composición de una traslación con una simetría axial es una simetría axial si el vector de traslación es ortogonal al eje de la simetría y en otro caso es una antitraslación:*
$$T_v S_l = \begin{cases} S_r, & \text{si } v \perp l, \text{ con } r = T_{\frac{v}{2}}(l) \\ A_{w,s} & \text{si } v \not\perp l, \text{ con } s = T_{\frac{v}{2}}(l) \end{cases},$$
$$S_l T_v = \begin{cases} S_r, & \text{si } v \perp l, \text{ con } r = T_{-\frac{v}{2}}(l) \\ A_{w,s} & \text{si } v \not\perp l, \text{ con } s = T_{-\frac{v}{2}}(l) \end{cases}.$$
El vector w de la antitraslación se obtiene como el cateto en la dirección de l del triángulo rectángulo de hipotenusa el vector v. Notar que r y s son paralelos a l.
4. *La composición de dos simetrías centrales es una traslación: $C_p C_q = T_{2(p-q)}$.*
5. *Si R_1 y R_2 son dos rotaciones con centros distintos, entonces $R_1 R_2 R_1^{-1} R_2^{-1}$ es una traslación.* □

El siguiente teorema permite determinar todos los grupos de friso.

Teorema 4.4. *Si G es un grupo de friso con $\mathcal{T}_G = \langle T_v \rangle$, entonces G es uno de los siguientes:*

- $\langle T_v \rangle = \{T_{nv} : \ n \in \mathbb{Z}\}$,
- $\langle T_v, C_p \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{C_{p+\frac{n}{2}v} : \ n \in \mathbb{Z}\right\}$,
- $\langle T_v, S_l \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \{A_{nv,l} : \ n \in \mathbb{Z}\} \cup \{S_l\}, \quad l \parallel v,$
- $\left\langle A_{\frac{v}{2},l} \right\rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{A_{\left(n+\frac{1}{2}\right)v,l} : \ n \in \mathbb{Z}\right\}$,
- $\langle T_v, S_r \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \{S_{r_n} : \ n \in \mathbb{Z}\}$, $\quad r \perp v, \ r_n = T_{\frac{n}{2}v}(r), \ \forall n,$
- $\langle T_v, S_r, S_l \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{C_{p+\frac{n}{2}v} : \ n \in \mathbb{Z}\right\} \cup \{S_{r_n} : \ n \in \mathbb{Z}\} \cup$ $\cup \{A_{nv,l} : \ n \in \mathbb{Z}\} \cup \{S_l\},$
  $l \parallel v, \ r \perp v, \ p = l \cap r, \ r_n = T_{\frac{n}{2}v}(r), \ \forall n,$
- $\left\langle S_r, C_q, A_{\frac{v}{2},l} \right\rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{C_{q+\frac{n}{2}v} : \ n \in \mathbb{Z}\right\} \cup \{S_{r_n} : \ n \in \mathbb{Z}\} \cup$ $\cup \left\{A_{\left(n+\frac{1}{2}\right)v,l} : \ n \in \mathbb{Z}\right\}$,
  $l \parallel v, \ r \perp v, \ p = l \cap r, \ q = T_{\frac{v}{4}}(p), \ r_n = T_{\frac{n}{2}v}(r), \ \forall n.$

Dem. Es un ejercicio el verificar que los grupos de la lista son grupos de friso. A continuación probaremos que todo grupo de friso G, con $\mathcal{T}_G = \langle T_v \rangle$, coincide con alguno de los anteriores.

Sea l_0 la recta por el origen paralela a v, r_0 la recta por el origen perpendicular a v y C la simetría central de centro en el origen. Veamos que $\tilde{G}$ es uno de los siguientes:

$$\tilde{G} = \{\mathrm{Id}\}, \quad \tilde{G} = \{\mathrm{Id}, C\}, \quad \tilde{G} = \{\mathrm{Id}, S_{l_0}\},$$

$$\tilde{G} = \{\mathrm{Id}, S_{r_0}\}, \quad \tilde{G} = \{\mathrm{Id}, S_{l_0}, S_{r_0}, C\}.$$

Si $M \in \tilde{G}$, entonces M deja fijo al origen y también sabemos que deja invariante $L_G = \mathbb{Z}v$. Esta última condición implica que M deja fijo a l_0 y al dejar fijo al origen las únicas posibilidades para M son $\mathrm{Id}, S_{l_0}, S_{r_0}, C$. Como $\tilde{G}$ es un grupo y $S_{l_0}S_{r_0} = S_{r_0}S_{l_0} = C$, vemos que las únicas posibilidades para $\tilde{G}$ son las anteriores.

Veamos ahora cómo las distintas opciones para $\tilde{G}$ dan lugar a los G descritos arriba. En lo que sigue aplicaremos reiteradamente la proposición 4.2.

Caso 1: $\tilde{G} = \{\mathrm{Id}\}$.

Lo único que hay en G son traslaciones, luego $G = \mathcal{T}_G = \langle T_v \rangle$.

Caso 2: $\tilde{G} = \{\mathrm{Id}, C\}$.

Como $C \in \tilde{G}$, entonces existe un punto p tal que $C_p \in G$. Luego $\langle T_v, C_p \rangle \subset G$. Notar que aplicando la proposición 4.3 obtenemos

$$\langle T_v, C_p \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \left\{C_{p+\frac{n}{2}v} : \ n \in \mathbb{Z}\right\}.$$

Probaremos $\langle T_v, C_p \rangle = G$. Observar que la proposición 4.2 nos dice que dada la forma de $\tilde{G}$, en G solo pueden haber traslaciones y simetrías centrales. Las traslaciones están en $\mathcal{T}_G = \langle T_v \rangle \subset \langle T_v, C_p \rangle$. Si $C_q \in G$, entonces $G \ni C_q C_p = T_{2(q-p)}$, luego existe $n \in \mathbb{Z}$ tal que $2(q-p) = nv$ y $C_q = C_{p+\frac{n}{2}v} \in \langle T_v, C_p \rangle$. Esto completa la prueba de $G = \langle T_v, C_p \rangle$.

Caso 3: $\tilde{G} = \{\mathrm{Id}, S_{l_0}\}$.

Como $S_{l_0} \in \tilde{G}$, entonces G contiene una simetría axial S_l o una antitraslación $A_{u,l}G$, con l paralelo a v.

Caso 3.A: Existe l recta paralela a v tal que $S_l \in G$.

Observar que $A_{v,l} = T_v S_l \in G$, luego

$$\langle T_v, S_l \rangle = \{T_{nv} : \ n \in \mathbb{Z}\} \cup \{A_{nv,l} : \ n \in \mathbb{Z}\} \cup \{S_l\}$$

Sea $M \in G \setminus \mathcal{T}_G$. Luego M solo puede ser una simetría axial o antitraslación de vector paralelo a v. En el primer caso, si $M = S_{l'}$ con $l' \neq l$, entonces $T_w = S_l S_{l'} \in G$ siendo w un vector no nulo perpendicular a v, lo cual es imposible; luego $M = S_l \in \langle T_v, S_l \rangle$.

En el segundo caso, sea $M = A_{u,l'} \in G$ con l' paralela a l. Si fuese $l' \neq l$, entonces $T_w = S_l A_{u,l'} \in G$ siendo w un vector no colineal con v; como esto es imposible deducimos $l = l'$. Luego $M = A_{u,l} \in G$ y por lo tanto $T_u = S_l A_{u,l} \in G$. Esto implica que existe $n \in \mathbb{Z}$ tal que $u = nv$, luego $M = A_{nv,l} \in \langle T_v, S_l \rangle$.

Caso 3.B: Existe l recta paralela a v y un vector u tal que $A_{u,l} \in G$.

Entonces $T_{2u} = \left(A_{u,l'}\right)^2 \in G$. Esto implica que existe $n \in \mathbb{Z}$ tal que $2u = nv$. Acá tenemos dos posibilidades, n es par o impar. Si $n = 2m$, con $m \in \mathbb{Z}$, entonces $u = mv$, $S_l = A_{u,l} T_{-mv} \in G$ y estamos en el caso 3.A.

Si $n = 2m + 1$, con $m \in \mathbb{Z}$, entonces $u = \left(m + \frac{1}{2}\right) v$ y $A_{\frac{v}{2},l} = A_{u,l} T_{-mv} \in G$. Probaremos que en este caso $G = \left\langle A_{\frac{v}{2},l} \right\rangle$. Observar que $\left(A_{\frac{v}{2},l}\right)^2 = T_v$, luego

$$\left\langle A_{\frac{v}{2},l} \right\rangle = \{T_{nv} :\ n \in \mathbb{Z}\} \cup \left\{ A_{\left(n+\frac{1}{2}\right)v,l} :\ n \in \mathbb{Z} \right\}.$$

Sea $M \in G \setminus \mathcal{T}_G$. Luego M solo puede ser una simetría axial o antitraslación de eje paralelo a l. En el primer caso, si $M = S_{l'}$, entonces $T_w = A_{\frac{v}{2},l} S_{l'} \in G$ siendo $w = \frac{v}{2}$ si $l = l'$ o un vector no colineal con v si $l' \neq l$; luego M no puede ser una simetría axial. Supongamos $M = A_{u,l'} \in G$ con l' paralela a l. No puede ser $l' \neq l$, porque en ese caso $T_w = A_{u,l'} A_{\frac{v}{2},l} \in G$ con w no colineal con v. Luego $M = A_{u,l} \in G$. Como antes tenemos $T_{2u} = \left(A_{u,l'}\right)^2 \in G$, luego existe $n \in \mathbb{Z}$ tal que $2u = nv$. Si n es par entonces G contendría una simetría axial lo cual ya vimos que no es posible, luego n es impar. Si $n = 2m + 1$, entonces $M = A_{u,l} = A_{\left(m+\frac{1}{2}\right)v,l} \in \left\langle A_{\frac{v}{2},l} \right\rangle$.

Caso 4: $\tilde{G} = \{\mathrm{Id}, S_{r_0}\}$.

Como $S_{r_0} \in \tilde{G}$, entonces existe r recta perpendicular a v tal que $S_r \in \tilde{G}$. Observar que $T_v S_r = S_{r'}$, siendo $r' = T_{\frac{v}{2}}(r)$, luego

$$\langle T_v, S_r \rangle = \{T_{nv} :\ n \in \mathbb{Z}\} \cup \{S_{r_n} :\ n \in \mathbb{Z}\},\ r_n = T_{\frac{n}{2}v}(r),\ \forall n.$$

Sea $M \in G \setminus \mathcal{T}_G$. Luego M solo puede ser una simetría axial o antitraslación de eje r' paralelo a r. Si $M = A_{u,r'} \in G$ con $u \neq 0$, entonces $T_{2u} = \left(A_{u,r'}\right)^2 \in G$ con u ortogonal a v, lo cual es imposible. Luego $M = S_{r'}$. Si r' no estuviese en $\{r_n = T_{\frac{n}{2}v}(r),\ \forall n\}$, entonces existiría un entero n tal que r' estaría entre r_n y r_{n+1}, y por lo tanto distaría de una de estas rectas en menos de $\frac{1}{2}v$; esto implicaría que G contiene una traslación $T_w \neq \mathrm{Id}$ con $\|w\| < \|v\|$, lo cual es imposible. Luego existe n tal que $r' = r_n$ y por lo tanto $M = S_{r_n} \in \langle T_v, S_r \rangle$.

Caso 5: $\tilde{G} = \{\mathrm{Id}, S_{l_0}, S_{r_0}, C\}$. Por el caso 4, sabemos que existe r recta perpendicular a v tal que $S_r \in \tilde{G}$ y por el caso 3 sabemos que hay dos posibilidades: $S_l \in G$ o $A_{\frac{v}{2},l} \in G$, con l paralela a v.

Caso 5.A: Existen r recta perpendicular a v y l recta paralela a v tales que $S_r, S_l \in G$.

Sea $C_p = S_r S_l \in G$. Observar que de los casos 2, 3A y 4 se deduce

$$\begin{aligned}\langle T_v, S_r, S_l \rangle &= \{T_{nv} :\ n \in \mathbb{Z}\} \cup \left\{C_{p+\frac{n}{2}v} :\ n \in \mathbb{Z}\right\} \cup \{S_{r_n} :\ n \in \mathbb{Z}\} \cup \\ &\quad \cup \{A_{nv,l} :\ n \in \mathbb{Z}\} \cup \{S_l\}, \quad r_n = T_{\frac{n}{2}v}(r),\ \forall n.\end{aligned}$$

Probaremos $G = \langle T_v, S_r, S_l \rangle$. Si $M \in G \setminus \mathcal{T}_G$, necesariamente M es uno de los siguientes

$$C_q, \quad S_{l'}, \quad A_{u,l'}, \quad S_{r'}, \quad A_{u,r'},$$

siendo l' paralela a v, r' perpendicular a v, q un punto y u un vector. Estudiemos cada una de estas posibilidades:

- **:** Si $M = C_q$, la prueba del caso 2 implica $M \in \langle T_v, C_p \rangle \subset \langle T_v, S_r, S_l \rangle$.
- **:** Si $M = S_{l'}$, la prueba del caso 3.A implica que $l' = l$ y $M = S_l \in \langle T_v, S_r, S_l \rangle$.
- **:** Si $M = A_{u,l'}$, la prueba del caso 3.A implica que $l' = l$ y $M = A_{nv,l} \in \langle T_v, S_r, S_l \rangle$.
- **:** Si $M = S_{r'}$, la prueba del caso 4 implica que $M \in \langle T_v, S_r \rangle \subset \langle T_v, S_r, S_l \rangle$.
- **:** Si $M = A_{u,r'}$, la prueba del caso 4 implica que esta opción no es posible.

Caso 5.B: Existen r recta perpendicular a v y l recta paralela a v tales que $S_r, A_{\frac{v}{2},l} \in G$.

Si p es la intersección de l con r, entonces $C_q = S_r A_{\frac{v}{2},l} \in G$, siendo $q = T_{\frac{v}{4}}(p)$. Observar que por lo que vimos en los casos 2, 3.B y 4, es

$$\begin{aligned}\langle S_r,\ A_{\frac{v}{2},l},\ C_q\rangle \ &= \ \{T_{nv} :\ n \in \mathbb{Z}\} \cup \left\{C_{q+\frac{n}{2}v} :\ n \in \mathbb{Z}\right\} \cup \{S_{r_n} :\ n \in \mathbb{Z}\} \cup \\ &\cup \left\{A_{\left(n+\frac{1}{2}\right)v,l} :\ n \in \mathbb{Z}\right\}, \quad r_n = T_{\frac{n}{2}v}(r),\ \forall n.\end{aligned}$$

Probaremos $G = \langle S_r,\ A_{\frac{v}{2},l},\ C_q\rangle$. Si $M \in G \setminus \mathcal{T}_G$, necesariamente M es uno de los siguientes

$$C_{q'}, \quad S_{l'}, \quad A_{u,l'}, \quad S_{r'}, \quad A_{u,r'},$$

siendo l' paralela a v, r' perpendicular a v, q' un punto y u un vector. Estudiemos cada una de estas posibilidades:

- : Si $M = C_{q'}$, la prueba del caso 2 implica $M \in \langle T_v, C_q\rangle \subset \langle S_r,\ A_{\frac{v}{2},l},\ C_q\rangle$.
- : Si $M = S_{l'}$, la prueba del caso 3.B implica que esta opción no es posible.
- : Si $M = A_{u,l'}$, la prueba del caso 3.B implica que $l' = l$ y $M = A_{\left(n+\frac{1}{2}\right)v,l} \in \langle S_r,\ A_{\frac{v}{2},l},\ C_q\rangle$.
- : Si $M = S_{r'}$, la prueba del caso 4 implica que $M \in \langle T_v, S_r\rangle \subset \langle S_r,\ A_{\frac{v}{2},l},\ C_q\rangle$.
- : Si $M = A_{u,r'}$, la prueba del caso 4 implica que esta opción no es posible. $\square$

Referencias

[1] M. A. Armstrong. *Basic Topology.* Ed. Springer

[2] G. Toth. *Glimpses of Algebra and Geometry, Second Edition.* Ed. Springer.

[3] H. Weyl. *Symmetry.* Princeton University Press.

GRAPH COLORING PROBLEMS

GUILLERMO DURÁN

1. The Four-Color problem

The Four-Color Conjecture was settled in the XIX century:

`Every map can be colored using at most four colors in such a way that adjacent regions (i.e. those sharing a common boundary segment, not just a point) receive different colors.`

1.1. **In terms of graphs...** Clearly a graph can be constructed from any map, the regions being represented by the vertices of the graph and two vertices being joined by an edge if the regions corresponding to the vertices are adjacent.

The resulting graph is **planar**, that is, it can be drawn in the plane without any edges crossing.

So, the Four-Color Conjecture asks if the vertices of a planar graph can be colored with at most 4 colors so that no two adjacent vertices use the same color.

1.2. **Example.**

1.3. **History.**

The Four-Color Conjecture first seems to have been formulated by Francis Guthrie. He was a student at University College London where he studied under Augusts De Morgan.

After graduating from London he studied law but some years later his brother Frederick Guthrie had become a student of De Morgan. Francis Guthrie showed his brother some results he had been trying to prove about the coloring of maps and asked Frederick to ask De Morgan about them.

Guthrie

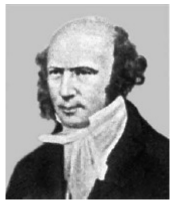
De Morgan

De Morgan was unable to give an answer but, on 23 October 1852, the same day he was asked the question, he wrote a letter to Sir William Hamilton in Dublin:

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact - and do not yet. He says that if a figure be anyhow divided and the compartments differently colored so that figures with any portion of common boundary line are differently colored - *four colors may be wanted, but not more* - the following is the case in which four colors are wanted. Query cannot a necessity for five or more be invented. ... If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did...

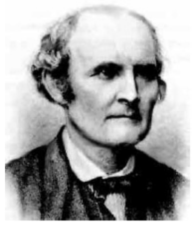
Hamilton replied on 26 October 1852 (showing the efficiency of both himself and the postal service):

> I am not likely to attempt your quaternion of colors very soon.



Hamilton

1.4. **First attempts.**



Cayley

The first published reference is found in Arthur Cayley's, *On the colorings of maps*, Proc. Royal Geographical Society 1, 259–261, 1879.

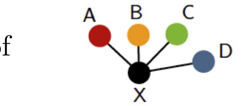
On 17 July 1879 Alfred Bray Kempe announced in Nature that he had a proof of the Four-Color Conjecture. Kempe was a London barrister who had studied mathematics under Cayley at Cambridge and devoted some of his time to mathematics throughout his life. At Cayley's suggestion Kempe submitted the Theorem to the American Journal of Mathematics where it was published in the ends of 1879.

Kempes

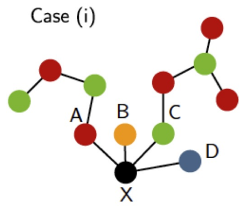
Kempe

1.5. **Idea of Kempe's proof.**

Kempe used an argument known as the method of *Kempe chains.*

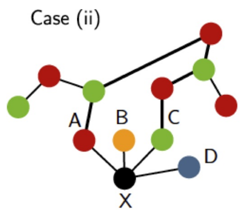


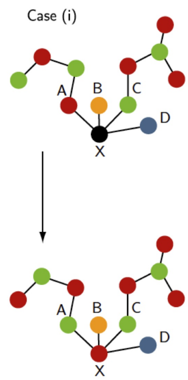
If we have a map in which every region is colored red, green, blue or yellow except one, say X. If this final region X is not surrounded by regions of all four colors there is a color left for X. Hence suppose that regions of all four colors surround X.



If X is surrounded by regions A, B, C, D in order, colored red, yellow, green and blue then there are two cases to consider.

(i) There is no chain of adjacent regions from A to C alternately colored red and green.
(ii) There is a chain of adjacent regions from A to C alternately colored red and green.

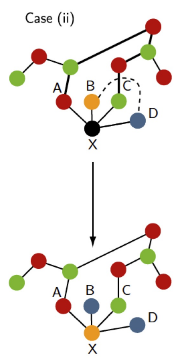




Cases:

(i) There is no chain of adjacent regions from A to C alternately colored red and green.
(ii) There is a chain of adjacent regions from A to C alternately colored red and green.

If (i) holds there is no problem. Change A to green, and then interchange the color of the red/green regions in the chain joining A. Since C is not in the chain it remains green and there is now no red region adjacent to X. Color X red.



If (ii) holds then there can be no chain of yellow/blue adjacent regions from B to D. [It could not cross the chain of red/green regions.] Hence property (ii) holds for B and D and we change colors as above.

The Four-Color Theorem returned to being the Four-Color Conjecture in 1890.

Percy John Heawood, a lecturer at Durham England, published a paper called *Map coloring theorem.* In it he states that his aim is "`...rather destructive than constructive, for it will be shown that there is a defect in the now apparently recognised proof...`".

Heawood

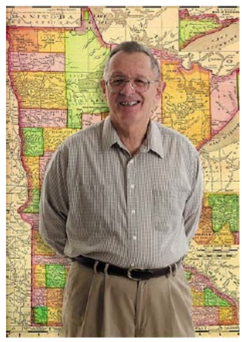
Although Heawood showed that Kempe's proof was wrong he did prove that every map can be 5-colored in this paper.

Exercise 1.1. *Using Kempe's ideas, prove that every map can be 5-colored.*
Hint: *Every planar graph has at least one vertex of degree at most 5.*

2. The proofs

It was not until 1976 that the four-color conjecture was finally proven by Kenneth Appel and Wolfgang Haken at the University of Illinois. They were assisted in some algorithmic work by John Koch.

- K. Appel and W. Haken, *Every planar map is four colorable. Part I. Discharging*, Illinois J. Math. 21 (1977), 429–490.
- K. Appel, W. Haken and J. Koch, *Every planar map is four colorable. Part II. Reducibility*, Illinois J. Math. 21 (1977), 491–567.



Appel

2.1. **Idea of the proof.** If the four-color conjecture were false, there would be at least one map with the smallest possible number of regions that requires five colors. The proof showed that such a minimal counterexample cannot exist through the use of two technical concepts:

- An unavoidable set contains regions such that every map must have at least one region from this collection.
- A reducible configuration is an arrangement of countries that cannot occur in a minimal counterexample. If a map contains a reducible configuration, and the rest of the map can be colored with four colors, then the entire map can be colored with four colors and so this map is not minimal.

Using different mathematical rules and procedures, Appel and Haken found an unavoidable set of reducible configurations, thus proving that a minimal counterexample to the four-color conjecture could not exist.

Their proof reduced the infinitude of possible maps to 1,936 reducible configurations (later reduced to 1,476) which had to be checked one by one by computer.

However, the unavoidability part of the proof was over 500 pages of hand written counter-counter-examples (these graph colorings were verified by Haken's son!). The computer program ran for hundreds of hours.

But most of the researchers thought that there were two reasons why the Appel-Haken proof was not completely satisfactory.

- Part of the Appel-Haken proof uses a computer, and cannot be verified by hand, and
- Even the part that is supposedly hand-checkable is extraordinarily complicated and tedious, and no one has verified it in its entirety.

Ten years ago, another proof:

- N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, *The four color theorem*, J. Combin. Theory Ser. B. 70 (1997), 2–44.
- N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, *A new proof of the four color theorem*, Electron. Res. Announc. Amer. Math. Soc. 2 (1996), 17–25 (electronic).


Robertson


Sanders


Seymour


Thomas

2.2. **Outline of the proof.** The basic idea of the proof is the same as Appel and Haken's. The authors exhibit a set of 633 "configurations", and prove each of them is "reducible". Recall, that this is a technical concept that implies that no configuration with this property can appear in a minimal counterexample to the Four-Color Theorem. It has been known since 1913 that every minimal counterexample to the Four-Color Theorem should be a special structure, called "internally 6-connected triangulation".

In the second part of the proof they prove that at least one of the 633 configurations appears in every internally 6-connected planar triangulation. This is called proving unavoidability, and here the method used differs from that of Appel and Haken. The first part of proof needs a computer. The second part can be checked by hand in a few months, or, using a computer, it can be verified in about 20 minutes.

2.3. **Why is this proof "better"?** The unavoidable set has size 633 as opposed to the 1476 member set of Appel and Haken, and the second part of the proof uses only about 30 rules, instead of the 300+ of Appel and Haken (and by computer can be verified in about 20 minutes against hundred of hours of the other proof).

At December 2004 in a scientific meeting in France, a joint group between people by Microsoft Research in England and INRIA in France announced the verification of the Robertson et al. proof by formulating the problem in the equational logic program Coq and confirming the validity of each of its steps (Devlin 2005, Knight 2005).

But in both cases (Appel and Haken, and Robertson et al.), the 'proofs' are not proofs in the traditional sense, because they contain steps that can never be verified by humans. Up today, a traditional mathematical proof is not known for the Four-Color Theorem.

3. Some basic concepts about computational complexity

- A *problem* is a general question to be answered, usually possessing several *parameters*, whose values are left unspecified.
- A problem is described by giving:
  (1) A general description of all its parameters.
  (2) A statement of what properties the answer (or *solution*) is required to satisfy.
- The difficulty of a problem is related to its structure and the length of the instance to be considered. This length is given by one or two parameters, for example, in the graph coloring problem, the number of vertices of the graph.
- In order to know the complexity of an algorithm we need to calculate the number of elementary arithmetic operations that the algorithm does to solve a given problem. This number is a function of the length of the instance.
- We say that a problem is in **P** if there exists an algorithm of polynomial complexity to solve it (the number of those operations is always upper bounded by a polynomial function in n, the input length).

3.1. **NP-completeness theory.**

- It is applied to *decision problems*, problems whose answer is "YES" or "NOT" (but it is easy to see that this theory has several consequences on optimization problems).
- For example, the decision problem related to the graph coloring problem is the following: *"Given a graph G and an integer number k, is there a valid coloring with at most k colors?"*
- A decision problem π consists of a set D_π of instances and a subset $Y_\pi \subseteq D_\pi$ whose answer is "YES".

- A problem $\pi \in \mathbf{NP}$ if there exists a polynomial certificate to verify an instance of "YES" (this is, if I can verify in polynomial time that an instance of "YES" is right).
- So, it is not difficult to see that P $\subseteq$ NP.
- **Open Conjecture: P $\neq$ NP.**

3.2. **Polynomial reduction.**

- Let π and π' be two decision problems. We say that $f : D_{\pi'} \to D_\pi$ is a *polynomial reduction* of π' in π if f can be computed in polynomial time and for every $d \in D_{\pi'}$, $d \in Y_{\pi'} \Leftrightarrow f(d) \in Y_\pi$. Notation: $\pi' \preccurlyeq \pi$.
- Note that if $\pi'' \preccurlyeq \pi'$ and $\pi' \preccurlyeq \pi$ then $\pi'' \preccurlyeq \pi$, because the composition of two polynomial reductions is a polynomial reduction.

3.3. **NP-complete problems.**

- A problem π is *NP-complete* if:
  (1) $\pi \in$ NP.
  (2) For every $\pi' \in$ NP, $\pi' \preccurlyeq \pi$.
- If a problem π verifies condition 2., we say that π is *NP-hard* (it is so "difficult" as all the problems in NP).

3.4. **P $\neq$ NP? or P = NP?.**

- **If there is a problem $\pi \in$ NP-c $\cap$ P, then P=NP.**
  - If $\pi \in$ NP-c $\cap$ P, there is a polynomial time algorithm to solve π, because π is in P. On the other hand, as $\pi \in$ NP-c, for every $\pi' \in$ NP, $\pi' \preccurlyeq \pi$.
  - Let π' be in NP. We have to use the polynomial reduction which transforms instances of π' in instances of π, and then the polynomial time algorithm which solves π. It is easy to see that we obtain a polynomial time algorithm to solve π'.
- It is known any problem neither in NP-c $\cap$ P, nor in NP $\setminus$ P (in this last case, it would be proved that P $\neq$ NP).

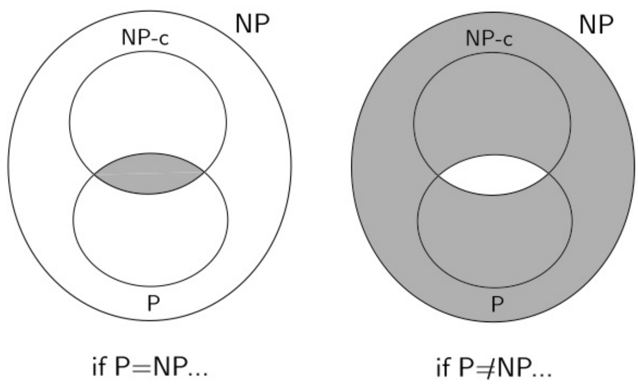
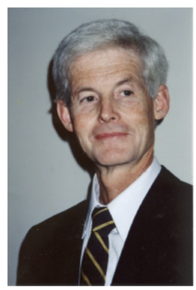


FIGURE 1. Inclusions between the classes

3.5. NP-completeness proofs.

Theorem 3.1 (Cook's Theorem (1971)). *SAT is NP-complete.*

The proof is direct: it is easy to see that SAT is in NP. Then, it is considered a general problem $\pi \in$ NP and a general instance $d \in D_\pi$. Using a polynomial non-deterministic Turing machine to solve π, it is generated in polynomial time a logic formula $\varphi_{\pi,d}$ such that $d \in Y_\pi$ if and only if $\varphi_{\pi,d}$ is satisfiable.



Cook

How do we have to do to prove that a problem is NP-complete?

Using Cook's Theorem, the standard technique to prove that a problem π is NP-complete uses the transitivity of $\preccurlyeq$, and consists in the following:

(1) Prove that π is in NP.

(2) Choose an appropriated problem π' belonging to NP-c.

(3) Build a polynomial reduction f of π' in π.

The second condition of the definition holds using the transitivity: let π'' be a problem in NP. As π' is NP-c, $\pi'' \preccurlyeq \pi'$. But it was proved that $\pi' \preccurlyeq \pi$, so $\pi'' \preccurlyeq \pi$.

Some famous problems in NP-c

- Traveling Salesman Problem (TSP)
- Graph coloring
- Integer Programming

4. Graph coloring

- A k-coloring of a graph G is an assignment of one color to each vertex of G such that no more than k colors are used and no two adjacent vertices receive the same color.
- A graph is called k-colorable iff it has a k-coloring.

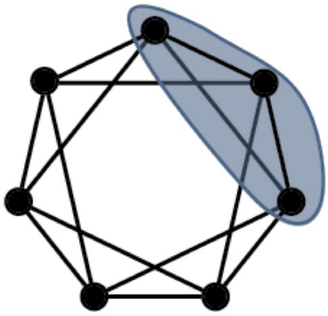
4-coloring

3-coloring

4.1. **Chromatic number.**

- A *clique* in a graph G is a complete subgraph maximal under inclusion. The cardinality of a maximum clique is denoted by $\omega(G)$.
- The *chromatic number* of a graph G is the smallest number k such that G is k-colorable, and it is denoted by $\chi(G)$. An obvious lower bound for $\chi(G)$ is $\omega(G)$:

$$\omega(G) \leq \chi(G) \quad \forall G.$$



$\omega = 3$

$\chi = 4$

4.2. **Applications.** The problem of coloring a graph has several applications such as scheduling, register allocation in compilers, frequency assignment in Mobile radios, etc.

Example: Examination schedule

Each student must take an examination in each of his/her courses. Let X be the set of different courses and let Y be the set of students. Since the examination is written, it is convenient that all students in a course take the examination at the same time. What is the minimum number of examination periods needed?

Exercise 4.1. *Model this problem as a coloring problem.*

4.3. **Computational complexity.** The graph k-colorability problem is the following:

- INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq V$.
- QUESTION: Is G k-colorable?

This problem is NP-complete (Karp, 1972), and remains NP-c for $k = 3$.

Exercise 4.2. *What happens for $k = 2$?*

Karp

4.4. **Planar graphs coloring.** For planar graphs the paper by Robertson et al. gives a quadratic algorithm to four-color planar graphs, an improvement over the quadric algorithm by Appel and Haken.

Exercise 4.3. *Does it mean that the k-colorability problem is polynomial for planar graphs?*

4.5. **Some easy properties about $\chi(G)$.**

- Let G be a graph with n vertices and $\overline{G}$ its complement. Then:
  - $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.
  - $\chi(G)\, \omega(\overline{G}) \geq n$
  - $\chi(G) + \omega(\overline{G}) \leq n + 1$
  - $\chi(G) + \chi(\overline{G}) \leq n + 1$

4.6. **Brooks' Theorem.**

Theorem 4.1. *Brooks' Theorem (1941) Let G be a connected graph. Then G is $\Delta(G)$-colorable, unless:*

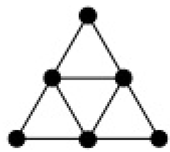
(1) $\Delta(G) \neq 2$, *and G is a* $\Delta(G) + 1$*-clique, or*
(2) $\Delta(G) = 2$, *and G is an odd cycle.*

4.7. **Graph coloring algorithms.** As it was said, it is not known a polynomial time algorithm to determine $\chi(G)$. Let us see the following no efficient algorithm (contraction-connection):

- Consider a graph G with two non-adjacent vertices a and b. The connection G_1 is obtained by joining the two non-adjacent vertices a and b with an edge. The contraction G_2 is obtained by shrinking $\{a, b\}$ into a single vertex $c(a, b)$ and by joining it to each neighbor in G of vertex a and of vertex b (and eliminating multiple edges).
- A coloring of G in which a and b have the same color yields a coloring of G_1. A coloring of G in which a and b have different colors yields a coloring of G_2.
- Repeat the operations of connection and contraction in each graph generated, until the resulting graphs are all cliques. If the smallest resulting clique is a k-clique, then $\chi(G) = k$.

4.8. **Graph coloring algorithms.**

Exercise 4.4. *Apply this method in the following graph*



4.9. **Chromatic polynomial.** The *chromatic polynomial* of a graph G is defined to be a function $P_G(k)$ that expresses for each integer k the number of distinct possible k-colorings for a graph G.

- **Example 1:** If G is a tree with n vertices, then:

$$P_G(k) = k(k-1)^{n-1}$$

- **Example 2:** If G is a n-clique, then:

$$P_G(k) = k(k-1)(k-2)\dots(k-n+1)$$

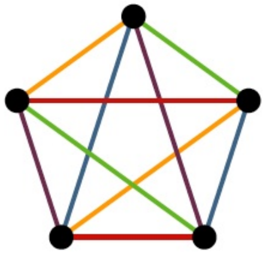
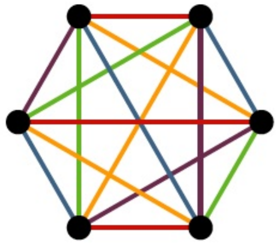
Property: $P_G(k) = P_{G_1}(k) + P_{G_2}(k)$, where G_1 and G_2 are the graphs defined in the connection-contraction algorithm.

Exercise 4.5. *Prove that the chromatic polynomial of a cycle C_n is:*

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$

4.10. **Chromatic index.**

- The *chromatic index* $\chi'(G)$ of a graph G is defined to be the smallest number of colors needed to color the edges of G so that no two adjacent edges have the same color.
- Clearly $\chi'(G) \geq \Delta(G)$, the maximum degree of G.
- A q-coloring of the edges of G is defined to be a partition of the edge set of G into q subsets that are matchings (a set of edges which do not share endpoints).

- **Property:** If G is a complete graph with n vertices, then
  - $\chi'(G) = n - 1$, if n is even
  - $\chi'(G) = n$, if n is odd
- **Vizing's Theorem (1964):** Let G be a graph, then

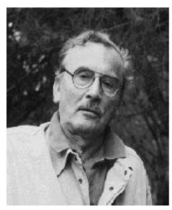
$$\Delta \leq \chi'(G) \leq \Delta(G) + 1.$$

- The problem of determining if there exists a $\Delta(G)$-coloring of a graph G is NP-complete (Holyer, 1981), even if the given graph is triangle-free with $\Delta(G) = 3$ (Koreas, 1997).

5. Perfect graphs

5.1. **Characterization and recognition.**

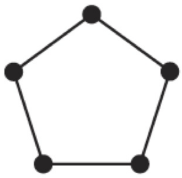
A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G (Berge, 1961).



Berge

Berge conjectured two statements:

(1) A graph is perfect if and only if its complement is perfect.
(2) A graph is perfect if and only if it contains neither induced odd cycle of length at least five nor its complement.



Theorem 5.1 (The Perfect Graph Theorem (Lóvasz, 1972; Fulkerson, 1973))**.** *A graph is perfect if and only if its complement is perfect.*

Lóvasz

Fulkerson

Exercise 5.1. *Prove that odd holes and their complements are not perfect.*

Theorem 5.2 (The Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour, Thomas, 2002))**.** *A graph is perfect if and only if it contains neither induced odd cycle of length at least five nor its complement.*

This work was published recently:

- Chudnovsky M., Robertson N., Seymour P. and Thomas R., *The Strong Perfect Graph Theorem*, Annals of Mathematics 164 (2006), 51–229.

Chudnovsky

Robertson

Seymour

Thomas

5.2. **Polynomial (but no efficient!) recognition.** The characterization by Chudnovsky et al. does not lead to a polynomial recognition of perfect graphs (the complexity of recognizing odd holes is open).

Cornuéjols

In 2002, two polynomial algorithms for recognizing perfect graphs were presented.

- *Recognizing Berge Graphs*, Chudnovsky and Seymour, 2002 (an $O(n^9)$ algorithm).
- *A Polynomial Algorithm for Recognizing Perfect Graphs*, Cornuéjols, Liu and Vušković, 2002 (an $O(n^{20})$ algorithm).

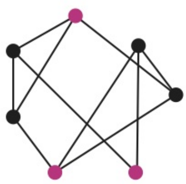
Vušković

In 2005, it was published the following joint work:

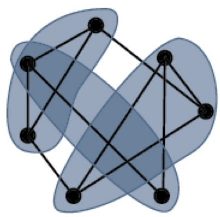
- Chudnovsky M., Cornuéjols G., Liu X., Seymour P. and Vušković K., *Recognizing Berge Graphs*, Combinatorica 25 (2005), 143–187.

5.3. **Another definition of perfect graphs.**

- An *independent set* (or *stable set*) in a graph G is a subset of pairwise non-adjacent vertices of G. The *stability number* $\alpha(G)$ is the cardinality of a maximum independent set of G.
- A *clique cover* of a graph G is a subset of cliques covering all the vertices of G. The *clique-covering number* $k(G)$ is the cardinality of a minimum clique cover of G.
- It is easy to see that $\alpha(G) = \omega(\overline{G})$ and $k(G) = \chi(\overline{G})$.
- So, by PGT: **A graph G is *perfect* when $\alpha(H) = k(H)$ for every induced subgraph H of G.**



$\alpha = 3$



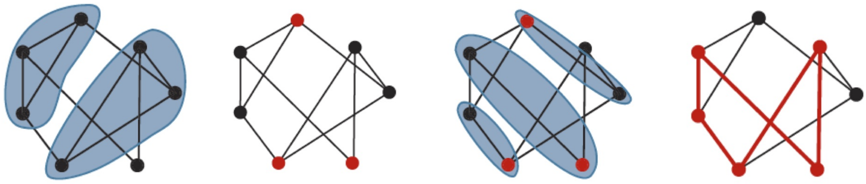
$k = 3$

5.4. **Variations of perfect graphs.**

5.4.1. *Clique-perfect graphs.*

- A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-independence number* $\alpha_c(G)$ is the size of a maximum clique-independent set of G.

- A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. The *clique-transversal number* $\tau_c(G)$ is the size of a minimum clique-transversal of G.
- Clearly, $\alpha_c(G) \leq \tau_c(G)$ for every graph G.
- A graph G is *clique-perfect* when $\alpha_c(H) = \tau_c(H)$ for every induced subgraph H of G.



- The terminology "clique-perfect" has been introduced by Guruswami and Pandu Rangan in 2000, but the equality of the parameters α_c and τ_c was previously studied by Berge in the seventies.

Guruswami

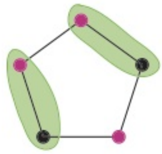
Pandu Rangan

- The complete list of minimal clique-imperfect graphs is still not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem.

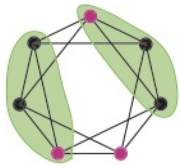
5.5. **Question: is there some relation between clique-perfect graphs and perfect graphs?**

- Odd holes $C_{2k+1}, k \geq 2$, are not clique-perfect: $\alpha_c(C_{2k+1}) = k$ and $\tau_c(C_{2k+1}) = k + 1$.

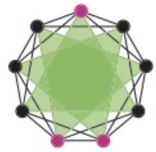
- Antiholes $\overline{C_n}, n \geq 5$, are clique-perfect if and only if $n \equiv 0(3)$ (Reed, 2000): $\tau_c(\overline{C_n}) = 3$ and $\alpha_c(\overline{C_n}) = 2$ or 3, being 3 only if n is divisible by three.



C_5

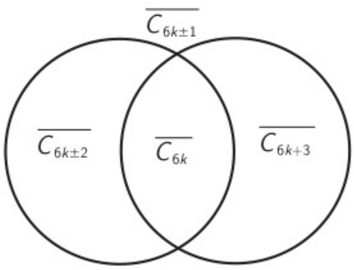


$\overline{C_7}$



$\overline{C_9}$

So we have the following scheme of relation between perfect graphs and clique-perfect graphs:



perfect clique-perfect

In several works, clique-perfect graphs have been characterized by a restricted list of forbidden induced subgraphs when the graph belongs to a certain class. Some of this characterizations lead to polynomial time recognition algorithms for clique-perfection within these classes.

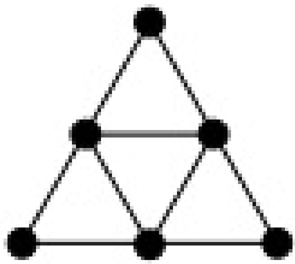
- J. Lehel and Z. Tuza, *Neighborhood perfect graphs*, Discrete Mathematics 61 (1986), 93–101.
- F. Bonomo, M. Chudnovsky and G.D., *Partial characterizations of clique-perfect graphs*, Electronic Notes in Discrete Mathematics 19 (2005), 95–101.
- F. Bonomo and G.D., *Characterization and recognition of Helly circular-arc clique-perfect graphs*, Electronic Notes in Discrete Mathematics 22 (2005), 147–150.

5.6. **Example: The characterization for line graphs.** Let H be a graph. Its line graph $L(H)$ is the intersection graph of the edges of H. A graph G is a *line graph* if there exists a graph H such that $G = L(H)$.

Theorem 5.3. *Let G be a line graph. Then the following statements are equivalent:*

(1) *No induced subgraph of G is and odd hole, or a pyramid.*
(2) *G is clique-perfect.*

(3) *G is perfect and it does not contain a pyramid.*



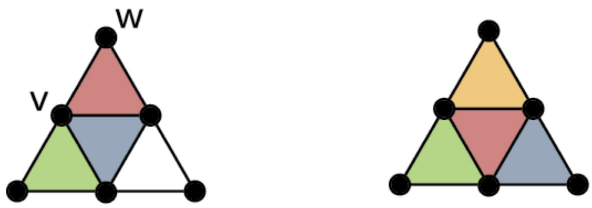
pyramid

Line graphs have polynomial time recognition (Lehot, 1974).

The recognition of clique-perfect line graphs can be reduced to the recognition of perfect graphs with no pyramid, which is solvable in polynomial time.

5.7. **Coordinated graphs.**

- Let v be a vertex of a graph G and $m(v)$ the number of cliques containing v.
- Let $M(G)$ be the maximum $m(v)$ for any v in G.
- Let $F(G)$ be the cardinality of a minimum partition of the cliques of G into clique-independent sets, that is, the smallest number of colors that can be assigned to the cliques of G so that intersecting cliques have different colors.
- Note that $F(G) \geq M(G)$ for any graph G.



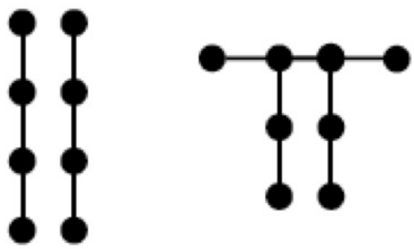
$m(v) = 3, m(w) = 1, M = 3$ $\quad\quad$ $F = 4$

- We say that a graph G is *coordinated* if $F(H) = M(H)$, for every induced subgraph H of G (this class of graph was defined by Bonomo, D. and Groshaus in 2002).

- **Property:** Coordinated graphs are perfect.
- The complete list of minimal non-coordinated graphs is still not known. Again, they have been characterized by a restricted list of forbidden induced subgraphs when the graph belongs to a certain class (Bonomo, D., Soulignac and Sueiro, 2006).

5.8. **Example: The characterization for complements of forests.** A forest is a graph with no cycles.

Theorem 5.4. *Let G be a complement of a forest. Then G is coordinated if and only if G contains neither $\overline{2P_4}$ nor $\overline{R}$ as induced subgraphs.*



$2P_4$ and R

This theorem leads to a linear time recognition of coordinated graphs if the given graph is a complement of a forest.

The general recognition of coordinated graphs is NP-hard (Soulignac and Sueiro, 2006).

5.9. **Some subclasses of perfect graphs.**

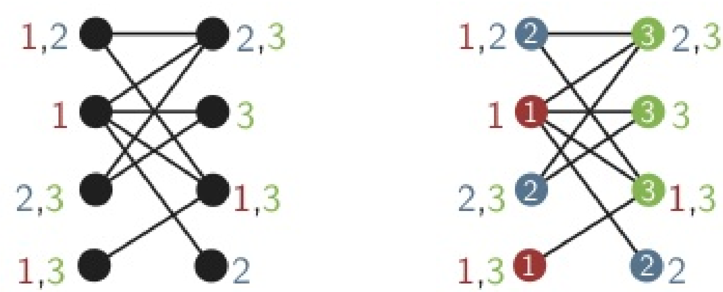
- A graph is an *interval graph* if it is the intersection graph of a set of intervals over the real line. A *unit interval graph* is the intersection graph of a set of intervals of length one.
- A *split graph* is a graph whose vertex set can be partitioned into a complete graph K and a stable set S. A split graph is said to be *complete* if its edge set includes all possible edges between K and S.
- A *bipartite graph* is a graph whose vertex set can be partitioned into two independent sets V_1 and V_2. A bipartite graph is said to be *complete* if its edge set includes all possible edges between V_1 and V_2.

- A *cograph* is a graph with no induced P_4.
- The *line graph* of a graph is the intersection graph of its edges. Line graphs of bipartite graphs are perfect.
- A graph is *distance-hereditary* if the distance between any two vertices in a connected induced subgraph containing both is the same as in the original graph.
- A graph is a *block graph* if it is connected and every maximal 2-connected component is complete.

6. Extensions of the coloring problem

6.1. **The list-coloring problem.** In order to take into account particular constraints arising in practical settings, more elaborate models of vertex coloring have been defined in the literature. One of such generalized models is the *list-coloring problem*, which considers a prespecified set of available colors for each vertex.

- Given a graph G and a finite list $L(v) \subseteq \mathbb{N}$ for each vertex $v \in V$, the list-coloring problem asks for a *list-coloring* of G, i.e., a coloring f such that $f(v) \in L(v)$ for every $v \in V$.



- Many classes of graphs where the vertex coloring problem is polynomially solvable are known, the most prominent being the class of perfect graphs [Grötschel-Lovász-Schrijver, 1981].
- Meanwhile, the list-coloring problem is NP-complete for perfect graphs, and is also NP-complete for many subclasses of perfect graphs, including split graphs, interval graphs, and bipartite graphs.
- Trees and complete graphs are two classes of graphs where the list-coloring problem can be solved in polynomial time. In the first case it can be solved using dynamic programming techniques

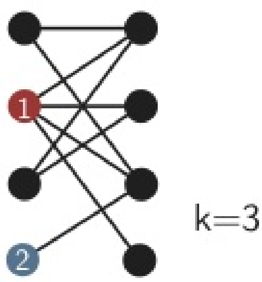
[Jansen-Scheffler, 1997]. In the second case, the problem can be reduced to the maximum matching problem in bipartite graphs.

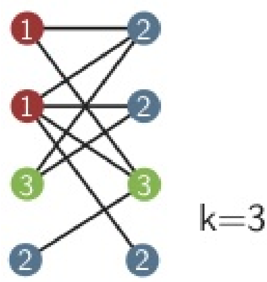
We are going to explore the complexity boundary between vertex coloring and list-coloring in different subclasses of perfect graphs. The goal is to analyze the computational complexity of coloring problems lying “between” (from a computational complexity viewpoint) these two problems.

6.2. **The precoloring extension problem.** Some particular cases of list-coloring have been studied.

- The *precoloring extension* (PrExt) problem takes as input a graph $G = (V, E)$, a subset $W \subseteq V$, a coloring f' of W, and a natural number k, and consists in deciding whether G admits a k-coloring f such that $f(v) = f'(v)$ for every $v \in W$ or not [Biro-Hujter-Tuza, 1992].

In other words, a prespecified vertex subset is colored beforehand, and our task is to extend this partial coloring to a valid k-coloring of the whole graph.

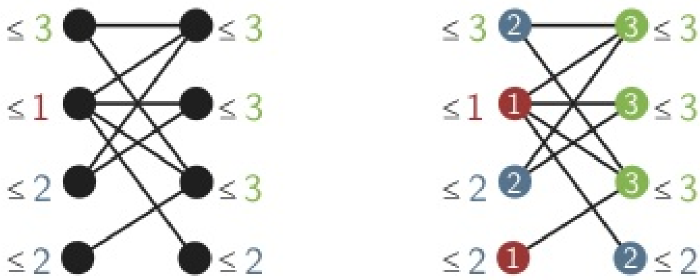




6.3. **The μ-coloring problem.** Another particular case of the list-coloring problem is the following.

- Given a graph G and a function $\mu : V \to \mathbb{N}$, G is *μ-colorable* if there exists a coloring f of G such that $f(v) \leq \mu(v)$ for every $v \in V$ [Bonomo-Cecowski, 2005].

This model arises in the context of classroom allocation to courses, where each course must be assigned a classroom which is large enough so it fits the students taking the course.

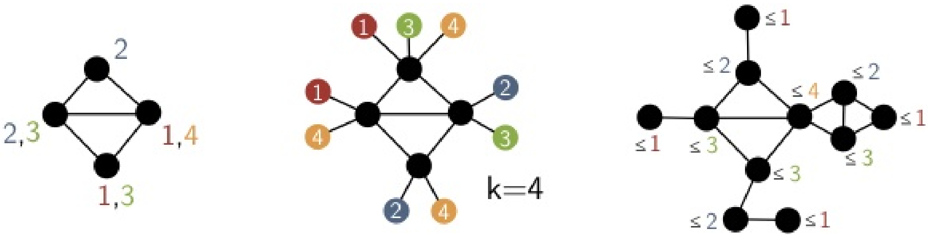


6.4. **The (γ, μ)-coloring problem.** A new variation of this problem is the following (Bonomo, D., Marenco, 2006).

- Given a graph G and functions $\gamma, \mu : V \to \mathbb{N}$ such that $\gamma(v) \leq \mu(v)$ for every $v \in V$, we say that G is (γ, μ)-*colorable* if there exists a coloring f of G such that $\gamma(v) \leq f(v) \leq \mu(v)$ for every $v \in V$.

- The classical vertex coloring problem is clearly a special case of μ-coloring and precoloring extension, which in turn are special cases of (γ, μ)-coloring.
- Furthermore, (γ, μ)-coloring is a particular case of list-coloring.
- These observations imply that all the problems in this hierarchy are polynomially solvable in those graph classes where list-coloring is polynomial and, on the other hand, all the problems are NP-complete in those graph classes where vertex coloring is NP-complete.

6.5. **General results.** Since all the problems are NP-complete in the general case, there are also polynomial-time reductions from list-coloring to precoloring extension and μ-coloring. An example is shown in the figure, where we can see a list-coloring instance and its corresponding precoloring extension and μ-coloring instances.



These reductions involve changes in the graph, but are closed within some graph classes. This fact allows us to prove the following general results.

Theorem 6.1. *Let $\mathcal{F}$ be a family of graphs with minimum degree at least two. Then list-coloring, (γ, μ)-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.*

Theorem 6.2. *Let $\mathcal{F}$ be a family of graphs satisfying the following property: for every graph G in $\mathcal{F}$, no connected component of G is complete, and for every cutpoint v of G, no connected component of $G \setminus v$ is complete. Then list-coloring, (γ, μ)-coloring, μ-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.*

Since odd holes and antiholes satisfy the conditions of the theorems above, these theorems are applicable for many subclasses of perfect graphs.

6.6. **Review of computational complexities.**

6.6.1. *Review: complexity table for coloring problems.*

Class	coloring	PrExt	μ-col.	(γ, μ)-col.	list-col.
Complete bipartite	P	P	P	P	NP-c
Bipartite	P	NP-c	NP-c	NP-c	NP-c
Cographs	P	P	P	?	NP-c
Distance-hereditary	P	NP-c	NP-c	NP-c	NP-c
Interval	P	NP-c	NP-c	NP-c	NP-c
Unit interval	P	NP-c	?	NP-c	NP-c
Complete split	P	P	P	P	NP-c
Split	P	P	NP-c	NP-c	NP-c
Line of $K_{n,n}$	P	NP-c	NP-c	NP-c	NP-c
Complement of bipartite	P	P	?	?	NP-c
Block	P	P	P	P	P

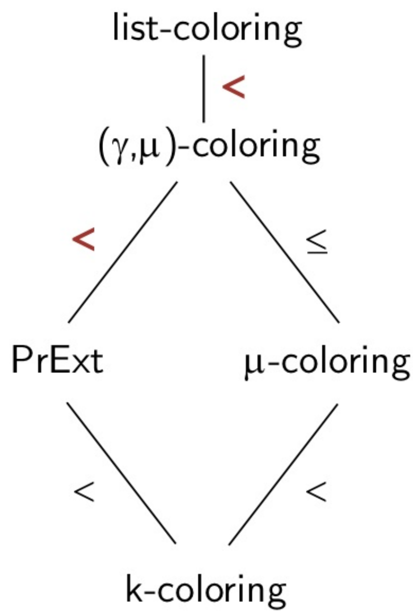
"NP-c": NP-complete problem, "P": polynomial problem, "?": open problem.

As this table shows, unless P = NP, μ-coloring and precoloring extension are strictly more difficult than vertex coloring, list-coloring is strictly more difficult than (γ, μ)-coloring and (γ, μ)-coloring is strictly more difficult than precoloring extension.

It remains as an open problem to know if there exists any class of graphs such that (γ, μ)-coloring is NP-complete and μ-coloring can be solved in polynomial time. Among the classes considered here, the candidate classes are *cographs*, *unit interval* and *complement of bipartite*.

For *split* graphs, precoloring extension can be solved in polynomial time, whereas μ-coloring is NP-complete. It remains as an open problem to find a class of graphs where the converse holds. Among the classes considered here, the candidate class is *unit interval*.

6.6.2. *Review: hierarchy of coloring problems.*



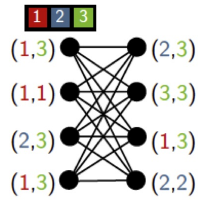
6.7. **Some proofs.**

6.7.1. *(γ, μ)-coloring is polynomial for complete bipartite graphs.* **Proof:** The following is a combinatorial algorithm that solves (γ, μ)-coloring in polynomial time for complete bipartite graphs.

Let $G = (V, E)$ be a complete bipartite graph, with bipartition $V_1 \cup V_2$, and let $\gamma, \mu : V \to \mathbb{N}$ such that $\gamma(v) \le \mu(v)$ for every $v \in V$.

We have to consider two cases:

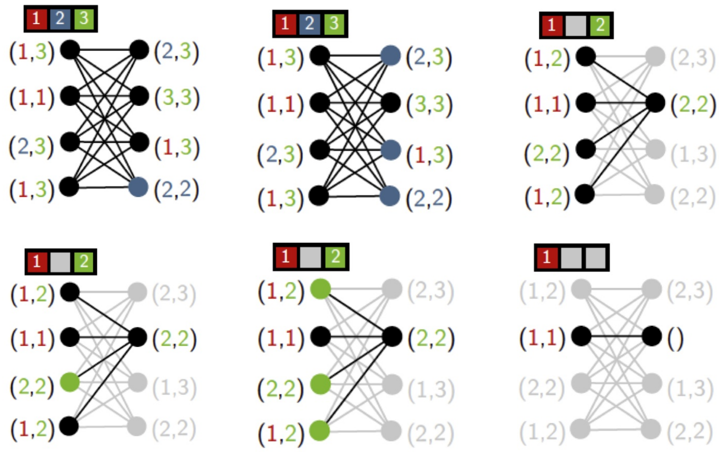
(i) There exists a vertex v such that $\gamma(v) = \mu(v)$.
(ii) For every vertex v, $\gamma(v) < \mu(v)$.



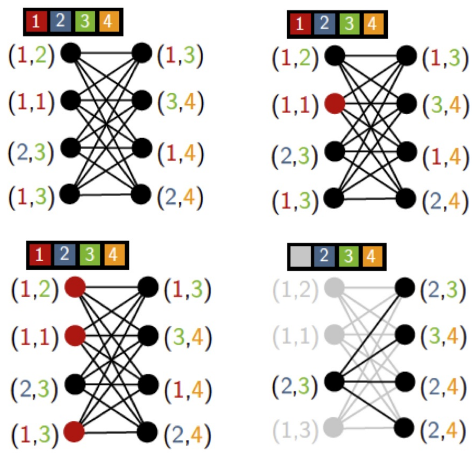
6.7.2. *Case (i):*

- If $\gamma(v) = \mu(v)$, the vertex v *must* be colored with color $\mu(v)$. Suppose $v \in V_2$. Since G is complete bipartite, no vertex of V_1 can use color $\mu(v)$.
- So, we can color with color $\mu(v)$ every vertex w of V_2 such that $\gamma(w) \le \mu(v) \le \mu(w)$ without affecting the feasibility of the problem.
- Then we remove those vertices and remove the color $\mu(v)$ from the universe of colors (we renumber the remaining colors so that they are still consecutive numbers).
- If some vertex of V_1 remains with no available color, the original graph was not (γ, μ)-colorable. Otherwise, we repeat this procedure until reaching either a coloring, or the non-colorability, or the case (ii).

Example 1:

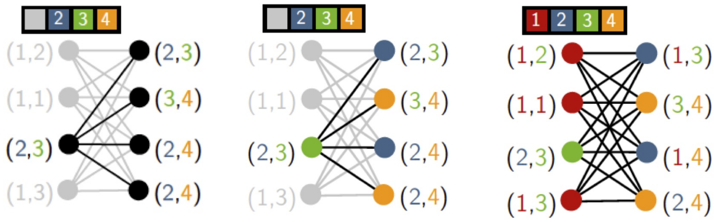


Example 2:



6.7.3. *Case (ii):*

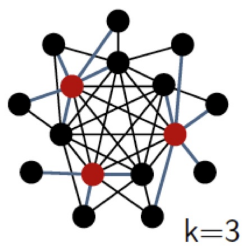
- If for every vertex v, $\gamma(v) < \mu(v)$, then every vertex has among its possible colors at least an odd color and an even color.
- So the graph is (γ, μ)-colorable, we can color the vertices of V_1 with odd colors and the vertices of V_2 with even colors.



□

6.7.4. *μ-coloring is NP-complete for split graphs.* **Proof:** It is used a reduction from the dominating set problem on split graphs, which is NP-complete (A. Bertossi, 1984).

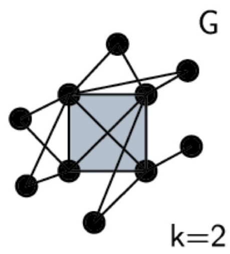
An instance of the dominating set problem on split graphs is given by a split graph G and an integer $k \geq 1$, and consists in deciding if there exists a subset D of $V(G)$, with $|D| \leq k$, and such that every vertex of $V(G)$ either belongs to D or has a neighbor in D. Such a set is called a *dominating set*.



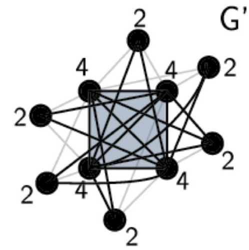
dominating set

Let G be a split graph and $k \geq 0$; $V(G) = K \cup I$, K is a complete and I is an independent set. We may assume G with no isolated vertices and $k \leq |K|$.

- We will construct a split graph G' and a function $\mu : V(G') \to \mathbb{N}$ such that G' is μ-colorable if and only if G admits a dominating set of cardinality at most k:
  - $V(G') = K \cup I$
  - K is a complete and I is an independent set in G'
  - for $v \in K$ and $w \in I$, $vw \in E(G')$ iff $vw \notin E(G)$
  - $\mu(v) = |K|$ for $v \in K$ and $\mu(w) = k$ for $w \in I$.

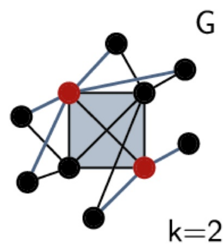


instance of split dominating set

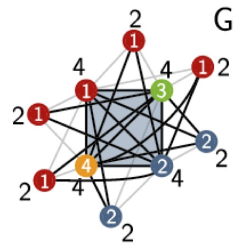


instance of split μ-coloring

- Suppose first that G admits a dominating set D with $|D| \leq k$. Since G has no isolated vertices, G admits such a set $D \subseteq K$.

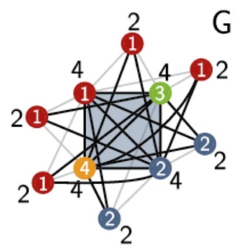


dominating set

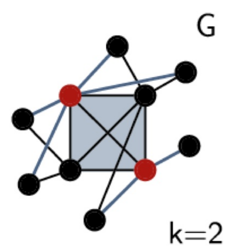


μ-coloring

- Let us define a μ-coloring of G' as follows:
  - color the vertices of D using different colors from 1 to $|D|$
  - color the remaining vertices of K using different colors from $|D| + 1$ to $|K|$
  - for each vertex w in I, choose w' in D such that $ww' \in E(G)$ and color w with the color used by w'.
- Suppose now that G' is μ-colorable, and let $c : V(G') \to \mathbb{N}$ be a μ-coloring of G'. Since $\mu(v) = |K|$ for every $v \in K$ and K is complete in G', it follows that $c(K) = \{1, \ldots, |K|\}$.



μ-coloring



dominating set

- Since $k \leq |K|$, for each vertex $w \in I$ there is a vertex $w' \in K$ such that $c(w) = c(w') \leq k$. Then $ww' \notin E(G')$, so $ww' \in E(G)$. Thus the set $\{v \in K : c(v) \leq k\}$ is a dominating set of G of size k.

$\square$

Acknowledgments: To Flavia Bonomo for her valuable help in the preparation of this course.

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ABOUT POINCARÉ-BIRKHOFF THEOREM

PATRICE LE CALVEZ

1. Introduction

In 1912, some weeks before his death, Henri Poincaré published a celebrated paper untitled *Sur un théorème de géométrie* published in *Rendiconti del Circolo Matematico di Palermo*. There probably no better way to introduce this paper than to copy out the introduction of Poincaré himself (see [P]).

Je n'ai jamais présenté au public un travail aussi inachevé ; je crois donc nécessaire d'expliquer en quelques mots les raisons qui m'ont déterminé à le publier, et d'abord celles qui m'avaient engagé à l'entreprendre. J'ai démontré, il y a longtemps déjà, l'existence des solutions périodiques du problème des trois corps ; le résultat laissait cependant encore à désirer ; car, si l'existence de chaque sorte de solution était établie pour les petites valeurs des masses, on ne voyait pas ce qui devait arriver pour des valeurs plus grandes, quelles étaient celles de ces solutions qui subsistaient et dans quel ordre elle disparaissaient. En réfléchissant à cette question, je me suis assuré que la réponse devait dépendre de l'exactitude ou de la fausseté d'un certain théorème de géométrie dont l'énoncé est très simple, du moins dans le cas du problème restreint et des problèmes de Dynamique où il n'y a que deux degrés de liberté.

J'ai donc été amené à rechercher si ce théorème est vrai ou faux, mais j'ai rencontré des difficultés auxquelles je ne m'attendais pas.

Date: November 14th, 2011.

J'ai été obligé d'envisager séparément un très grand nombre de cas particuliers ; mais les cas possibles sont trop nombreux pour que j'aie pu les étudier tous. J'ai reconnu l'exactitude du théorème dans tous ceux que j'ai traités. Pendant deux ans, je me suis efforcé sans succès, soit de trouver une démonstration générale, soit de découvrir un exemple où le théorème soit en défaut.

Ma conviction qu'il est toujours vrai s'affermissait de jour en jour, mais je restais incapable de l'asseoir sur des fondements solides.

Il me semble que dans ces conditions, je devrais m'abstenir de toute publication tant que je n'aurai pas résolu la question ; mais après les inutiles efforts que j'ai faits pendant de longs mois, il m'a paru que le plus sage était de laisser le problème mûrir, en m'en reposant durant quelques annés ; cela serait très bien si j'étais sûr de pouvoir le reprendre un jour ; mais à mon âge je ne puis en répondre. D'un autre côté, l'importance du sujet est trop grande (et je chercherai plus loin à la faire comprendre) et l'ensemble des résultas obtenus trop considérable déjà, pour que je me résigne à les laisser définitivement infructueux. Je puis espérer que les géomètres qui s'intéresseront à ce problème et qui seront sans doute plus heureux que moi, pourront en tirer quelque parti et s'en servir pour trouver la voie dans laquelle ils doivent se diriger.

Je pense que ces considérations suffisent me justifier.

Thus, Poincaré [P], conjectured the following : an area-preserving homeomorphism of the closed annulus that satisfies some "twist condition" admits at least two fixed points. He was able to prove it in many cases but unable to get a proof that works in the general case. However he was convinced of its validity and aware of the big importance of this result in celestial mechanics and in many other dynamical systems.

In 1913, Georges. D. Birkhoff [Bi1] gave a proof of this result. His proof used an index argument that was valid to find one fixed point but uncorrect to get the second one. A small modification of the argument was necessary and Birkhoff corrected this minor error in a paper [Bi3] published in 1925 (see also the well-detailed expository paper of Brown and Newman [BrowN]) where he made some generalizations: the area-preserving hypothesis was replaced by a topological intersection property, and the hypothesis about the invariance of the annulus was weakened. A generalization of the *Last Geometric Theorem of Poincaré*, more usually called *Poincaré-Birkhoff Theorem*, with a nicer intersection property, was obtained by Patricia Carter [Ca].

In this survey, we will begin by stating the original statement of the theorem as some other modificated versions. In the second section we will outline the origin of the problem: finding periodic orbits in the Plane Restricted Three Body Probem, and will give some examples of dynamical systems where the theorem (or one its generalized versions) may be applied, particulary to show the existence of periodic orbits in the study of non autonomous planar Hamiltonian systems.

In the third section we will outline the arguments of Poincaré and the proof of Birkhoff. We will give a complete proof based on the ideas of Birkhoff (see Le Calvez-Wang [LeW]). We will finish by explaing the fruitful ideas that John Franks has brought on the subject, based on Brouwer's theory of plane homeomorphismss.

In the last section we will talk alout the importance of Poincaré-Birkhoff Theorem in the development of symplectic geometry as the starting point of fixed points results for Hamiltonian isotopies on symplectic mainifolds.

2. Statement of the theorem

2.1. **Notations, homeomorphims of the annulus.** The Poincaré-Birkhoff Theorem is a fixed point theorem for some homeomorphisms of the "closed annulus". Let us begin by explaining what do we mean by a closed annulus. The simplest model is given by fixing two positve real numbers a and b satisfying $a < b$ and defining the set

$$\mathbf{A}_{a,b} = \{(u, v) \in \mathbf{R}^2 \mid a^2 \leq u^2 + v^2 \leq b^2\}.$$

All such sets are clearly homeomorphic and any topological space homeomorphic to $\mathbf{A}_{a,b}$ will be called a *closed annulus* in what follows. Taking the polar coordinates system on $\mathbf{A}_{a,b}$ gives us another model which will be more convenient to work on. Let us consider the quotient group $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ munished with the quotient topology, called the *one dimensional torus* or the *circle* (as it is homeomorphic to an euclidean circle). In the whole text, we will consider the following annulus $\mathbf{A} = \mathbf{T}^1 \times [0,1]$. The universal covering space of $\mathbf{A}$ is the strip $\widetilde{\mathbf{A}} = \mathbf{R} \times [0,1]$ and the covering projection is the map

$$\begin{aligned} \pi \ &: \widetilde{\mathbf{A}} \to \mathbf{A} \\ (x,y) &\mapsto (x + \mathbf{Z}, y). \end{aligned}$$

The group of covering transformations is generated by the translation

$$\begin{aligned} T \ &: \widetilde{\mathbf{A}} \to \widetilde{\mathbf{A}} \\ (x,y) &\mapsto (x+1, y). \end{aligned}$$

We will also consider the two projections

$$\begin{aligned} p_1 \ &: \widetilde{\mathbf{A}} \to \mathbf{R} \\ (x,y) &\mapsto x, \end{aligned}$$

and

$$\begin{aligned} p_2 \ &: \widetilde{\mathbf{A}} \to [0,1] \\ (x,y) &\mapsto y. \end{aligned}$$

We will be interested by homeomorphisms F of $\mathbf{A}$ that preserve the orientation and let invariant each boundary circle $\mathbf{T}^1 \times \{0\}$ and $\mathbf{T}^1 \times \{1\}$. It is a classical fact that such homeomorphisms are exactly the homeomorphisms F that are *isotopic to the identity* : there exists a family $(F_t)_{t\in[0,1]}$ of homeomorphisms of $\mathbf{A}$ such that :

- F_0 is the identity map;
- $F_1 = F$;
- the map $(t,z) \mapsto F_t(z)$ is continuous.

If F is such a homeomorphism, a *lift* of F to $\widetilde{\mathbf{A}}$ is a homeomorphism f of $\widetilde{\mathbf{A}}$ such that $F \circ \pi = \pi \circ f$. Lifts always exist and two lifts f and f' differ from an iterate of T: there exists $k \in \mathbf{Z}$ such that

$f' = T^k \circ f$. The homeomorphisms f of $\widetilde{\mathbf{A}}$ that lift a homeomorphism of $\mathbf{A}$ isotopic to the identity are characterized by the following, they satisfied the two properties :

- f preserves the orientation and let invariant each boundary line $\mathbf{R} \times \{0\}$ and $\mathbf{R} \times \{1\}$;
- for every $(x, y) \in \mathbf{R} \times [0, 1]$, one has $f(x + 1, y) = f(x, y) + (1, 0)$.

The second assertion tells us that f commutes with T. A consequence of this property is the following fact: if $z = (x, y)$ is a fixed point of f, then every point in the T-orbit of z, that means every point $(x + k, y)$, $k \in \mathbf{Z}$, is also a fixed point of f. All these fixed points project by π on the same point of $\mathbf{A}$, which is fixed by the homeomorphism F of $\mathbf{A}$ that is lifted by f.

The 2-form $dx \wedge dy$ defines an area form on $\mathbf{A}$ and so a finite measure. In what follows, we will say that F preserves the area if its preserves this measure. Equivalently, this means that any lift f of F preserves the Lebesgue measure on $\widetilde{\mathbf{A}}$.

2.2. **The statement.** Let us state now the Poincaré-Birkhoff Theorem :

Theorem 1 : *Let F be a homeomorphism of $\mathbf{A}$ isotopic to the identity and f a lift of F to $\widetilde{\mathbf{A}}$. We suppose that:*

i) *for every $x \in \mathbf{R}$, one has* $p_1 \circ f(x, 0) < x < p_1 \circ f(x, 1)$*;*

ii) *F preserves the area.*

Then f has at least two fixed points whose T-orbits are distinct.

2.2.1. *Remarks.*

- The two fixed points given by the theorem projects into two different fixed points of F.
- The condition **i)** is usually called the *boundary twist condition.* It is necessary because the map $f\ :\ (x,y)\mapsto(x+1/2,y)$ lifts a homeomorphism of $\mathbf{A}$ isotopic to the identity, satisfies **ii)** and is fixed point free.
- The condition **ii)** is also necessary because the map $(x,y)\mapsto(x+y-1/2,y^2)$ lifts a homeomorphism of $\mathbf{A}$ isotopic to the identity, satisfies **i)** and is fixed point free.
- Replacing f by f^{-1} gives us a version of the theorem where each sign $<$ is replaced by $>$ in the condition **i)**.

2.3. **Existence of periodic points.** We will explain now why the homeomorphism F of the theorem has infinitely many periodic points with period arbitrarily large.

Let F be a homeomorphism of $\mathbf{A}$ isotopic to the identity and f a lift to $\widetilde{\mathbf{A}}$. A *periodic point* of F is a point $z\in\mathbf{A}$ which is fixed by an iterate F^q of F, where $q\geq 1$. The smallest such integer is the *period* of z. The orbit of a periodic point z is a finite set O of cardinality equal to the period and every point in the orbit is also periodic with the same period. For every lift $\widetilde{z}\in\pi^{-1}(\{z\})$ of z, there exists an integer $p\in\mathbf{Z}$ such that $f^q(\widetilde{z})=T^p(\widetilde{z})$, because $\pi(f^q(\widetilde{z}))=F^q(z)=z$. This integer p does not depend neither on the choice of $\widetilde{z}$ nor on the choice of $z\in O$. One will say that p/q is the *rotation number* of z (or of O). Conversely, every point $\widetilde{z}\in\mathbf{A}$ such that there exists integers $q\geq 1$ and $p\in\mathbf{Z}$ with $f^q(\widetilde{z})=T^p(\widetilde{z})$, lifts a periodic point of rotation number p/q. One deduces that the period of this periodic point is q, in the case where p and q are relatively prime. Observe that taking the lift $T^r\circ f$ instead of f will transform a rotation number p/q into $p/q+r$.

The Poincaré-Birkhoff Theorem implies the following :

COROLLARY 2 : *Let F be a homeomorphism of* $\mathbf{A}$ *isotopic to the identity and f a lift of F to* $\widetilde{\mathbf{A}}$. *We suppose that*

i') *there exist two distinct real numbers ρ_0 and ρ_1 such that for every $x \in \mathbf{R}$, one has $f(x,0) = (x + \rho_0, 0)$ and $f(x,1) = (x + \rho_1, 1)$.*

ii) *F preserves the area.*

Then, every rational number $\rho = p/q$ (written in an irreducible way) in the open interval bounded by ρ_0 and ρ_1 is the rotation number of a periodic orbit of F of period q.

Proof. Let $\rho = p/q$ be a rational number written in an irreducible way in the open interval bounded by ρ_0 and ρ_1. The homeomorphism $f^q \circ T^{-p}$ is a lift of F^q that satisfies the hypothesis of the Poincaré-Birkhoff Theorem and therefore has at least a fixed point. Such a fixed point lifts a periodic point of F of period q and rotation number p/q. $\square$

2.3.1. *Remarks.*

• An obvious consequence is that F has infinitely many periodic orbits with period arbitrarily large.

• As a consequence, if f satisfies the hypotheses of Poincaré-Birkhoff Theorem and coincides with a translation on each boundary line, then F has infinitely many periodic orbits with period arbitrarily large. Let us explain now why this last statement is still valid if f satisfies the hypotheses of Poincaré-Birkhoff Theorem, without any additional condition (the details on what follows can be found in the Appendix). If f lifts a homeomorphism of $\mathbf{A}$ isotopic to the identity, then one can write

$$f(x,0) = (g_0(x), 0)), \;\; f(x,1) = (g_1(x), 1)$$

and get two increasing homeomorphisms g_0, g_1 of $\mathbf{R}$, satisfying

$$g_0(x+1) = g_0(x) + 1, \;\; g_1(x+1) = g_1(x) + 1.$$

An increasing homeomorphism g of $\mathbf{R}$ that satisfies $g(x+1) = g(x)+1$, for every $x \in \mathbf{R}$, is the lift of an orientation preserving homeomorphism G of $\mathbf{T}^1$. To such a homeomorphism is associated a *rotation number* $\rho(g)$, real number introduced by Poincaré, that measures the mean displacement of any orbit of g : for every $x \in \mathbf{R}$ and every $k \in \mathbf{Z}$, one has

$$-1 < g^k(x) - x - k\rho(g) < 1.$$

Among the properties satisfied by the rotation number, there is the following equality valid for every integers p and q:

$$\rho(g^q - p) = q\rho(g) - p.$$

Note also the following fundamental fact : $\rho(g)$ is equal to zero if and only if g has a fixed point; it is positive if and only if $g(x) > 0$ for every $x \in \mathbf{R}$; it is negative if and only if $g(x) < 0$ for every $x \in \mathbf{R}$. One deduces that the corollary 2 is still valid, with the same proof, if one replaces the assumption **i')** by the more general one:

i") the two rotation numbers $\rho_0 = \rho(g_0)$ and $\rho_1 = \rho(g_1)$ are distinct.

Observe now that this assumption is fullfilled if f satisfies the hypothesis of Poincaré-Birkhoff Theorem because $\rho_0 < 0 < \rho_1$.

2.4. **The intersection property.** We have seen that the map $(x, y) \mapsto (x + y - 1/2, y^2)$ lifts a homeomorphism isotopic to the identity, satisfies **i)** and is fixed point free. Observe that every horizontal circle $y = y_0$, $0 < y_0 < 1$, is disjoint from its image by F. It was observed by Birkhoff in his paper of 1925 [Bi3] that the area preserving assumption could be replaced by an intersection property (see also Kérekjártó [K]). More precisely :

Theorem 3 : *Let F be a homeomorphism of $\mathbf{A}$ isotopic to the identity and f a lift of F to $\widetilde{\mathbf{A}}$. We suppose that:*

i) *for every $x \in \mathbf{R}$, one has* $p_1 \circ f(x, 0) < x < p_1 \circ f(x, 1)$*;*

ii') *every* essential *loop of $\mathbf{A}$ (that means non homotopic to zero) intersects its image by F.*

Then f has at least one fixed point.

Of course if F preserves the area, or any other finite measure with total support, then it will satisfy the intersection property. This is also the case if F *has no wandering point* (this means that every non empty open subset meets at least one of its positive iterates) or if F preserves a measure that is finite on each sub-annulus $\mathbf{T}^1 \times [0, y_0]$, $y_0 \in (0,1)$. Observe that in all the mentioned cases, we have the follwing stronger property:

ii") every essential loop of $\mathbf{A}$ intersects its image by F in at least two points .

A result of Patricia Carter [Ca] asserts that one can replace the condition **ii)** by the condition **ii")** in the statement of Poincaré-Birkhoff Theorem: f has at least two fixed points whose T-orbits are distinct.

2.5. **Non invariant boundary.** Another fact, observed by Birkhoff in [Bi2] is that the invariance of the boundaries is not necessary. Fix two essential simple loops C_i, $i \in \{0,1\}$, in $\mathbf{T}^1 \times (0,+\infty)$ that project injectively onto $\mathbf{T}^1 \times \{0\}$ and denote by $\mathbf{A}_i$ the closed annulus bounded by $\mathbf{T}^1 \times \{0\}$ and C_i. Write $\widetilde{C}_i$ and $\widetilde{\mathbf{A}}_i$ for the respective preimages of C_i and $\mathbf{A}_i$ in the universal cover $\mathbf{R} \times [0,+\infty)$. Consider a homeomorphism $F\ : \mathbf{A}_0 \to \mathbf{A}_1$ that preserves the orientation and leaves invariant the circle $\mathbf{T}^1 \times \{0\}$, and a lift $f\ : \widetilde{\mathbf{A}}_0 \to \widetilde{\mathbf{A}}_1$ that satisfies the following twist condition :

$$z \in \mathbf{R} \times \{0\} \Rightarrow p_1 \circ f(z) < p_1(z), \;\; z \in \widetilde{C}_0 \Rightarrow p_1 \circ f(z) > p_1(z).$$

The first assertion below is due to Birkhoff and the second is the most general version of Carter's theorem [Ca].

THEOREM 4 : *If every essential simple loop in $\mathbf{A}_0$ meets its image by F, then f has at least one fixed point. If every essential simple loop in $\mathbf{A}_0$ meets its image by F in at least two points (for example if F preserves a finite measure with total support) then f has at least two fixed points whose T-orbits are distinct.*

Remark. It must be noticed that the theorem above is still valid if one only of the loops C_1 and C_2 projects injectively onto $\mathbf{T}^1 \times \{0\}$ (see for example Ding [Di] or Guillou [Gu1] for the argument) but it cannot be generalized to the situation where none of the loops projects injectively (see Martins-Ureña [MaU] or Le Calvez-Wang [LeW]).

One interesting consequence of Theorem 4 and the remark above is that it permits to state a version in the infinite annulus.

THEOREM 5 : *Let F be a homeomorphism of $\mathbf{T}^1 \times [0, +\infty[$ isotopic to the identity and f a lift of F to $\mathbf{R} \times [0, +\infty[$. We suppose that:*

i') *one has $\lim_{y\to+\infty} \min_{x\in\mathbf{R}}(p_1 \circ f(x, y) - x) = +\infty$;*

ii') *every essential loop of $\mathbf{T}^1 \times [0, +\infty[$ intersects its image by F.*

Then F has infinitely many fixed points. More precisely, if n is large enough, there exists at least one fixed point whose rotation number is n.

Proof Choose n_0 such that $p_1 \circ f(x, 0) - x - n_0 < 0$, for every $x \in \mathbf{R}$. For any $n \geq n_0$, one can find a real number M such that $p_1 \circ f(x, M) - x - n > 0$, for every $x \in \mathbf{R}$. One can apply Theorem 4 to the map $T^{-n} \circ f$: it must have at least one fixed point. □

2.5.1. *Remarks.*

• One may prove that each iterate F^q, $q \geq 2$, satisfies the intersection property **ii')** if it is the case for F. Applying the proof above to the map $f^q \circ T^{-p}$, one can show similarly that every rational number $\rho = p/q$ written in an irreducible way, larger that the rotation number of the map $x \mapsto f(x, 0)$, is the rotation number of a periodic orbit of F of period q.

• Of course, if one replaces the assumption **ii')** by the stronger assumption **ii")** in Theorem 5, one will get at least two fixed points whose rotation number are n, for n large enough.

• There exists also a version on the whole annulus that can be obtained using the arguments given in [Di] ot [**?**] (see also Addas-Zanata [Ad]). If one supposes that F is defined on $\mathbf{T}^1 \times \mathbf{R}$, satifies the assumption **ii')** written in $\mathbf{T}^1 \times \mathbf{R}$ and the assumption **i')** where we add the condition

$$\lim_{y \to -\infty} \max_{x \in \mathbf{R}} (p_1 \circ f(x, y) - x) = -\infty,$$

then for every integer k, there exists at least one fixed point whose rotation number is k.

3. Origin of the theorem, applications of the theorem

3.1. **Origin of the problem.** Originally, the Poincaré-Birkhoff Theorem comes from celestial mechanics. Following Newton's gravitation theory, one knows that n bodies in $\mathbf{R}^3$ of respective masses m_1, m_2, $\dots$, m_n, move following the differential equation

$$m_i \frac{d^2}{dt^2} \vec{x}_i = -G \sum_{j \neq i} m_i m_j \frac{\vec{x}_i - \vec{x}_j}{\|\vec{x}_i - \vec{x}_j\|^3},$$

where $\vec{x}_i \in \mathbf{R}^3$ is the position of the i-th body and G the gravitation constant. In the case where $n = 2$, one can integrate this system: the center of mass M moves under a rectilign motion of constant speed and the bodies describe conics with a focus at M in the frame whose origin is M. When $n \geq 3$ the system is no more integrable. The decisive contribution of Poincaré was the introduction of the qualitative study of this problem, in other words the study of the geometric structure of the orbits of this differential equation, and the introduction of tools which will be fundamental in what will become "dynamical systems".

One of the "simplest" case is the so called Planar Restricted Three Body Problem. We consider a planar two body problem, with the Sun (S) of mass μ and the Earth (E) of math $\nu < \mu$ that have a uniform circular motion around the center of mass. Now we consider a third body, the Moon (M) with a zero mass, that is assumed to move in the plane determined by the orbits of the two other bodies. The motion of the zero mass is studied in a coordinate system rotating around the center of mass so that the two other bodies are fixed in this new coordinate system. We identify this rotating plane to the complex plane $\mathbf{C}$ by supposing E fixed at the origin and S fixed at the point 1. Everything will be normalized in such a way that we will suppose that the gravitational constant G is equal to 1, that $\mu + \nu = 1$ and that the frequency of rotation of the rotating system is 1. The motion of M follows a Hamiltonian system with two degrees of freedom, that may be written

$$\begin{cases} x' = \dfrac{\partial H}{\partial \overline{y}} \\ \\ y' = -\dfrac{\partial H}{\partial \overline{x}} \end{cases}$$

where $x = x_1 + i\, x_2$ and $y = y_1 + i\, y_2$ are the position and the momentum of M respectively, and where the Hamiltonian is

$$H(x, y) = |y|^2 + i(\overline{x}y - x\overline{y}) - \frac{2\nu}{|x|} - \frac{2\mu}{|1+x|} - \mu(x + \overline{x}) + 2\mu.$$

For such a system, H is an integral of these equations: each orbit in the phase space (x, y) lies in an energy surface $H = a$. One can observe that if a is a large negative number, then the projection of the energy surface $H = a$ in the x space is made of three connected components: the so called *Hill's regions* that are neighborhood of S, E and ∞ respectively. Moreover as the energy approaches $-\infty$, the regions collapse onto the corresponding points.

The case studied by Poincaré was the case where M was moving in a Hill's region close to S. He began to study the limit case where $\mu = 1$ and $\nu = 0$. This is nothing but a two body problem in the rotating frame, it is an integrable (but not degenerate) system. Using a Levi-Civita transformation, Poincaré constructed on a given energy surface, a two dimensional annulus which is a *surface of section*. The boundary circles of the annulus are two closed orbits of the system and all other orbits in the energy surface must intersect transversally the annulus. The study of our three dimensional continuous dynamical system is reduced to the study of a two dimensional discrete dynamical system, that means the study of the iterates of a transformation of the annulus: take a point in the annulus and look at the next time the orbit passing through this point will come back in the annulus. It could be shown that the homeomorphism induced on the annulus preserves the area and rotates the points on the boundary circles at different speed. Having discussed the case where $\nu = 0$, Poincaré showed that for small values of ν, his construction could be naturally continued with an annular surface of section and an induced area preserving homeomorphism rotating the points on the boundary circles at different speed. Therefore a positive answer to the Poincaré-Birkhoff Theorem would imply the existence of infinitely many periodic orbits with arbitrarily large period on a negative energy surface, provided the mass of E is very small. The study of Poincaré was improved by Birkhoff [Bi2] and then by Conley [Co] in his thesis. Conley observed that the same phenomenon appears for every couple of masses, provided the energy is a sufficiently large negative number. In fact he studied the case where M was moving in a Hill's region close to E. The argument is not the same: if the energy is supposed to be a large negative number, then M will stay close to E, which implies that the attraction of E on M is much stronger that the attraction of S. Here again, the system appears to be a perturbation of a two body problem.

3.2. **Application to the dynamics of the convex billiard.** We will introduce here another mechanical system which is modelled by an area preserving homeomorphism of an annulus with a boundary twist condition. It is the example of the *convex billiard.* Let us consider a convex simple closed curve $\Gamma \subset \mathbf{R}^2$ of class C^k, $k \geq 2$. We want to describe the dynamics generated by the free motion of a point subject to the elastic reflection on the boundary. The orbits of such motion consist of straight line segments inside the curve joined at boundary points according to the rule that the angle of incidence equals the angle of reflection. It is a geodesic flow on a surface with boundary and can be seen as the limit case of the geodesic flow on a boundaryless subsurface of $\mathbf{R}^3$. There is a natural Poincaré section given by the intersections with the boundary and the dynamics can be described by a discrete dynamical system on $\mathbf{T}^1 \times [0,\pi]$ described as follows. We will suppose for conveniency that the length of Γ is 1 so that there is a parametrization

$$\begin{aligned} \gamma\ &: \mathbf{T}^1 \to \Gamma \\ &\qquad s \mapsto \gamma(s) \end{aligned}$$

by arc length. Fix $s \in \mathbf{T}^1$ and $\theta \in (0,\pi)$ and consider the line Δ passing through $\gamma(s)$ and directed by the vector v such that $\langle \gamma'(s), v\rangle = \theta$. This line will intersect Γ in another point $\gamma(s')$, $s' \neq s$, and the angle $\theta' = \langle v, \gamma'(s')\rangle$ belongs to $(0,\pi)$. The line Δ' passing through $\gamma(s')$ and directed by the vector v' such that $\langle \gamma'(s'), v'\rangle = \theta'$ is the line obtained from Δ by reflection on the curve at $\gamma(s')$. Therefore the dynamics of the billiard is decribed by the dynamics of $F\ : (s,\theta) \mapsto (s',\theta')$. Observe that this map can be extended to a homeomorphism of $\mathbf{T}^1 \times [0,\pi]$ that fixes each point of the boundary curves $\mathbf{T}^1 \times \{0\}$ and $\mathbf{T}^1 \times \{1\}$ and that its restriction to $\mathbf{T}^1 \times (0,\pi)$ is a diffeomorphism of class C^{k-1}. Indeed write $\delta = \{(s,s), s \in \mathbf{T}^1\}$ for the diagonal of $\mathbf{T}^1 \times \mathbf{T}^1$ and consider the two maps

$$\begin{aligned} G\ &: (\mathbf{T}^1 \times \mathbf{T}^1) \setminus \delta \to \mathbf{T}^1 \times (0,\pi) \\ &\qquad (s,s') \mapsto (s, \theta(s,s')) \end{aligned}$$

and

$$\begin{aligned} G'\ &: (\mathbf{T}^1 \times \mathbf{T}^1) \setminus \delta \to \mathbf{T}^1 \times (0,\pi) \\ &\qquad (s,s') \mapsto (s', \theta'(s,s')) \end{aligned}$$

where

$$\theta(s,s') = \langle \gamma'(s), \gamma(s') - \gamma(s)\rangle \ \text{ and } \ \theta'(s,s') = \langle \gamma(s') - \gamma(s), \gamma'(s')\rangle.$$

These maps, which are of class C^{k-1}, are diffeomorphisms. Indeed, one can prove that

$$\frac{\partial \theta}{\partial s'}(s,s') = \frac{\sin\theta'(s,s')}{\|\gamma(s')-\gamma(s)\|} > 0,$$

and

$$\frac{\partial \theta'}{\partial s}(s,s') = -\frac{\sin\theta'(s,s')}{\|\gamma(s')-\gamma(s)\|} < 0.$$

Observe now that the diffeomorphism $F = G' \circ G^{-1}$ of $\mathbf{T}^1 \times (0,\pi)$ extends to a homeomorphism of $\mathbf{T}^1 \times [0,\pi]$ that fixes the points on the boundary. Moreover the difference between the rotation number of the fixed points that are on $\mathbf{T}^1 \times \{\pi\}$ and the rotation numbers of the points that are on $\mathbf{T}^1 \times \{0\}$ is 1 (for any given lift). In fact if the curvature $\rho(s)$ never vanishes, F is a C^{k-1} diffeomorphism because

$$\frac{\partial \theta}{\partial s'}(s,s) = -\frac{\partial \theta'}{\partial s}(s,s) = \frac{\rho(s)}{2}.$$

Let us prove now that F preserves a finite measure. Let us define the following map

$$\begin{aligned} l \ : \mathbf{T}^1 \times \mathbf{T}^1 &\to \mathbf{R} \\ (s,s') &\mapsto \|\gamma(s')-\gamma(s)\|. \end{aligned}$$

Observe that l is differentiable and that

$$dl = \cos\theta'(s,s')ds' - \cos\theta(s,s')ds.$$

By writing $ddl = 0$ we deduce that F preserves the area form $\sin\theta\, ds \wedge d\theta$.

The map

$$\begin{aligned} H \ : \mathbf{T}^1 \times [0,\pi] &\to \mathbf{T}^1 \times [-1,1] \\ (s,\theta) &\mapsto (s,t) = (s, -\cos\theta) \end{aligned}$$

conjugates F to a homeomorphism F^* of $\mathbf{T}^1 \times [-1,1]$ that preserves the Lebesgue measure but won't be a diffeomorphism on the closed annulus even if the curvarture never vanishes. Corollary 2 gives us the following:

THEOREM 6 : *Let p and q be two integers relatively prime such that $0 < p < q$. Then there exists a periodic trajectory of the billiard with q segments $[z_i, z_{i+1}]$, $0 \leq i < q$, such that the q oriented subarcs $[z_i, z_{i+1}]_\Gamma$ of Γ cover the curve Γ exactly p times.*

3.3. **Applications to some differential equations.** The Poincaré-Birkhoff theorem can be applied to study certains differential equations of Hamiltonian type. Here is an example due to Jacobowitz (see [J1] and [J2] for the details, see also Dalbono-Rebelo [DaR]).

Let us consider the differential equation

$$(\mathrm{E}) : \quad x'' + f(x,t) = 0,$$

where $f : \mathbf{R}^2 \to \mathbf{R}$ is C^1. It can be written as the following system of equations

$$\begin{cases} x' = y \\ y' = -f(x,t) \end{cases}$$

that can be written

$$\begin{cases} x' = \dfrac{\partial H}{\partial y} \\ \\ y' = -\dfrac{\partial H}{\partial x}, \end{cases}$$

where

$$H(x,y,t) = \frac{1}{2} y^2 + \int_0^x f(u,t)\, du.$$

Let us suppose that

a) all maximal solutions are defined on $\mathbf{R}$,

b) f is periodic in t with period 1,

c) $xf(x,t) > 0$ for $x \neq 0$,

d) $\lim_{|x| \to +\infty} \min_{t \in \mathbf{R}} (f(x,t)/x) = +\infty$,

e) $f(0,t) = 0$ for every $t \in \mathbf{R}$.

One gets a family $(F_t)_{t\in\mathbf{R}}$ of C^1-diffeomorphisms of $\mathbf{R}^2$ defined by the following: for every point $(x,y)\in\mathbf{R}^2$, the map $t\mapsto F_t(x,y)$ is the solution of the system above with initial condition (x,y) at time $t=0$. The assumption **b)** implies that $F_{t+1}=F_t\circ F_1$ for every $t\in\mathbf{R}$. Thus, to study the system it is sufficient to study the dynamics of $F=F_1$. Each map F_t fixes the origin. Indeed the assumption **e)** implies that $x(t)=0$ is a solution of the equation (E). Taking a polar system of coordinates, one gets an isotopy $(F_t)_{t\in\mathbf{R}}$ defined on $\mathbf{T}^1\times[0,+\infty)$ with F_0 equal to identity. This family can be lifted to an isotopy $(f_t)_{t\in\mathbf{R}}$ on $\mathbf{R}\times[0,+\infty)$ with f_0 equal to the identity. Every periodic point of F of period q and rotation number p/q for the lift $f=f_1$ corresponds to a periodic solution of (E) of period q that vanishes exactly $2p$ times on the interval $[0,q)$ (because of condition **c)**. Because of condition **d)**, one can prove that F satisfies the condition **i')** of Theorem 5. It also satisfies the condition **ii")** because it is area preserving. Indeed the vector field $(y,-f(x,t))$ is Hamiltonian (it is divergence free). Applying Theorem 5, one gets :

Theorem 7 : *The equation $x''+f(x,t)=0$ has infinitely many periodic solutions of period* 1*. More precisely, if N is large enough, one can find a solution with $2N$ zeros in* $[0,1)$*. Moreover the equation has periodic solutions of period q with q arbitrarily large.*

3.4. **Applications to the dynamics of homeomorphisms of surfaces.** The Poincaré-Birkhoff Theorem has many applications to the study of homeomorphisms of surfaces. As an illustration, we give here a simple example:

Theorem 8 : *Let S be a closed orientable surface of genus $g\geq 1$ and F be an orientation preserving C^1 diffeomorphism of S that satisfies the following assumptions:*

i) *F admits a fixed point z such that the eigenvalues of $Df(z)$ are conjugate complex numbers $e^{\pm 2i\pi\alpha}$, where $\alpha\notin\mathbf{Q}$.*

ii) *F preserves a finite measure with total support.*

Then F has infinitely many periodic points with period arbitrarily large. In the case where F is isotopic to the identity a similar conclusion holds if we replace **i)** *by the weaker hypothesis*

i') *$DF(z)$ has no positive real eigenvalue.*

Proof Let us begin with the case where $g \geq 2$. One may identify the universal covering space of S with the Poincaré disk $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$. Fix a lift $\widetilde{z} \in \mathbf{D}$ of z. There is a unique lift $\widetilde{F}$ of F that fixes $\widetilde{z}$. The Nielsen-Thurston theory of homeomorphisms of surfaces tells us the following (see Miller [Mi] for example):

- there is a continuous extension of $\widetilde{F}$ to $\overline{\mathbf{D}} = \{z \in \mathbf{C} \mid |z| \leq 1\}$;

- the rotation number which is induced on $S^1 = \{z \in \mathbf{C} \mid |z| =\}$ is rational:

- the extension fixes every point of S^1 in the case where F is isotopic to the identity.

By hypothesis, $\widetilde{F}$ is a diffeomorphism of $\mathbf{D}$. So, one can blow up the point $\widetilde{z}$ and compactify $\overline{D} \setminus \{\widetilde{z}\}$ by adding a circle Σ at $\widetilde{z}$, via a polar system of coordinates, in such a way that the restricted homeomorphism on $\overline{D} \setminus \{\widetilde{z}\}$ will extend continuously and induce on Σ the natural action of $DF(z)$ on the space of half lines. If F satisfies **i)**, the restriction of $\widetilde{F}$ on Σ is conjugate to a rotation of angle $\alpha + \mathbf{Z}$ on $\mathbf{T}^1$; if F satisfies **i')** it is fixed point free and therefore has a non zero rotation number. In both cases we have constructed a homeomorphism of a compact annulus, isotopic to the identity and having two distinct rotation numbers on the boundary circles. The condition **ii)** implies that $\widetilde{F}$ preserves a measure of total support that is finite on every compact of $\mathbf{D}$. Therefore, the intersection property **ii")** of section 1 is satisfied, which implies that $\widetilde{F}$ has infinitely many periodic orbits with period arbitrarily large. These periodic orbits projects onto periodic points of F.

In the case where $g = 1$, everything works similarly replacing the Poincaré disk by the Euclidean plane. $\square$

4. About the proofs

4.1. **Poincaré's ideas.**

Let us begin this section by recalling what is Poincaré's attempt to prove the theorem. He is interested in the case where the map is smooth and satisfies additional properties (we will now say *generic properties*) and in this situation the existence of one fixed point implies the existence of the second one (for index reasons that we will describe later in this section). More precisely he supposes that the set

$$\widetilde{\Xi} = \{(x, y) \in \mathbf{R} \times (0, 1) \,|\, p_2 \circ f(x, y)) = y\}$$

is a smooth one dimensional manifold with connected components of three types:

- simple loops;
- arcs that are invariant by the covering transformation T and that project onto essential simple loops of $\mathbf{A}$;
- arcs with two ends on the boundary lines.

Moreover he supposes that the set $\widetilde{\Lambda}$ of points on $\widetilde{\Xi}$ where the tangent is horizontal projects onto a finite set Λ of $\mathbf{A}$. He gives a proof by contradiction by supposing that f is fixed point free. His goal is to construct either a simply connected domain $U \subset \widetilde{\mathbf{A}}$ bounded by a simple loop Γ, which is forward or backward invariant by f but not globally invariant, or an annular domain $U \subset \mathbf{A}$ bounded by an essential simple loop Γ, which is forward or backward invariant by F but not globally invariant. The first assertion contredicts both the absence of a fixed point and the fact that f preserves the area, the second case contredicts this last hypothesis.

Each connected component of $\widetilde{\Xi}$ is either a *right component* (one has $p_1 \circ f(x, y)) > x$ on it) or a *left component* (one has $p_1 \circ f(x, y)) < x$). Each connected component of $\widetilde{\mathbf{A}} \setminus \widetilde{\Xi}$ is either a *up component* (one has $p_2 \circ f(x, y)) > y$) or a *down component* (one has $p_2 \circ f(x, y)) < y$). From any point $z \in \widetilde{\Lambda}$ one can draw two horizontal segments until they they reach again Ξ. One obtains a natural graph made of $\widetilde{\Xi}$ and of the union of all these segments. There is a natural orientation of the edges of this graph which satisfies the following: every simple loop $\Gamma \subset \widetilde{\mathbf{A}}$ obtained by concatenating such oriented arcs bounds a simply connected domain which is forward or backward invariant by f but not globally invariant; every essential simple loop $\Gamma \subset \mathbf{A}$ obtained by concatenating the projections in $\mathbf{A}$ of such oriented arcs bounds an annular domain which is forward or backward invariant by F but not globally invariant. Consequently, the problem is reduced to try to avoid the *culs de sac*, that means the vertices that correspond to sinks or sources of the oriented graph. In every situation he can draw, Poincaré is able to construct such a curve, moreover he can give general proofs for certain classes of maps. His problem is to give a proof that works for any situation. The ideas of Poincaré have been brought to fruition, with some modifications, by Golé and Hall [GoHa].

We will illustrate the ideas of Poincaré in the simplest cases. We will make a slight modification by replacing the previous set $\widetilde{\Xi}$ by

$$\widetilde{\Xi} = \{(x, y) \in \mathbf{R} \times (0, 1) \,|\, p_1 \circ f(x, y)) = x\}$$

and the set $\widetilde{\Lambda}$ by the points where the tangent is vertical. We set $\Xi = \pi(\widetilde{\Xi})$ and $\Lambda = \pi(\widetilde{\Lambda})$. There are some natural reasons to do so, one of them is the fact that generically the set Ξ is a compact one-dimensional manifold with no boundary. With this modification, there are two types of components of Ξ, the up and down components, and two types of components of $\widetilde{\mathbf{A}} \setminus \widetilde{\Xi}$, the right and left components. Observe that the component of $\widetilde{\mathbf{A}} \setminus \widetilde{\Xi}$ that contains $\mathbf{R} \times \{0\}$ is always a left component and that the component that contains $\mathbf{R} \times \{1\}$ is a right component. In particular, the set Ξ separates the two boundary circles and contains an essential connected component. We will suppose that Ξ is reduced to such a curve (or equivalently that Ξ is connected) and we will construct an essential loop $\Gamma \subset \mathbf{A}$ that bounds an annular domain which is forward or backward invariant by F but not globally invariant.

The simplest case, is the case where Ξ is a graph. Its image is also a graph that is disjoint from Ξ. As Ξ is up or down, the loop $\Gamma = \Xi$ works. Suppose now that there are exactly two points z_0, z_1 in Λ. From each point $z_i \in \Lambda$, draw the vertical segment $[z_i, z_i']$ that reaches Ξ in its other extremity z_i'. Observe that the vertical vector $(0,1)$ is an eigenvector of the derivative $Df(z_i)$, $i \in (0,1\}$. More precisely, it is associate to a positive eigenvalue for both derivatives (the transation case) or to a negative eigenvalue (the reflection case). One can make the following observations:

- in the translation case, there exists a unique $i \in \{0,1\}$ such that Γ can be constructed by replacing one the sub-arcs of Ξ delimited by z_i and z_i' by the segment $[z_i, z_i']$;
- in the rotation case, the same occurs but the loop $\Gamma = \Xi$ itself also works.

4.2. **Birkhoff's ideas.**

Let us recall now the ideas of Birkhoff. We suppose that F and f satisfy the hypotheses of Theorem 1 and want to prove that the set $\mathrm{Fix}_*(F)$ of fixed points of F that are lifted to fixed points of f is neither empty, nor reduced to a point. Let us consider the vector field $\widetilde{\xi} : z \mapsto f(z) - z$ defined on $\widetilde{A}$. It is invariant by the covering automorphism T and therefore lifts a vector field ξ on $\mathbf{A}$ which vanishes exactly on $\mathrm{Fix}_*(F)$. To prove that this vector vanishes in at least two points, we will use an index argument. If γ is a path in $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$, one may define the *variation of angle*

$$i_\gamma \widetilde{\xi} = \int_{\widetilde{\xi} \circ \gamma} d\theta$$

where

$$d\theta = \frac{1}{2\pi} \frac{xdy - ydx}{x^2 + y^2}$$

is the usual polar form on $\mathbf{R}^2 \setminus \{0\}$. The form $d\theta$ being closed can be integrated on any path in $\mathbf{R}^2 \setminus \{0\}$, smooth or not. If z is an isolated zero of $\widetilde{\xi}$, one can define the *Poincaré index* $i(\widetilde{\xi}, z)$ of $\widetilde{\xi}$ at z (also called the *Lefschetz index* of f at z): it is equal to $i_\Gamma\widetilde{\xi}$, where is Γ is any simple loop sufficiently close to z that counter-clockwise surrounds this point. This integer is equal to the Poincaré index $i(\xi, \pi(z))$ of ξ at $\pi(z)$. In the case where $\mathrm{Fix}_*(F)$ is finite, then $i(\xi, z)$ is defined at every point $z \in \mathrm{Fix}_*(F)$ and one can apply the Poincaré-Hopf formula:

$$\sum_{z\in\mathrm{Fix}_*(F)} i(\xi, z) = \chi(\mathbf{A}) = 0,$$

the integer $\chi(\mathbf{A})$ being the Euler characteristic of $\mathbf{A}$. Consequently, to prove that $\mathrm{Fix}_*(F)$ contains at least two points, it is sufficient to prove that there exists at least one point $z \in \mathrm{Fix}_*(F)$ such that $i(\xi, z) \neq 0$, or equivalently that there exists at least one point $z \in \mathrm{Fix}(f)$ such that $i(\widetilde{\xi}, z) \neq 0$, To prove this, it is sufficient to construct a loop Γ in $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$ such that $i_\Gamma\widetilde{\xi} \neq 0$. Indeed, if $\mathrm{Fix}(f)$ is discrete, then

$$i_\Gamma\widetilde{\xi} = \sum_{z\in\mathrm{Fix}(f)} i(\widetilde{\xi}, z)\ i_\Gamma\widetilde{\xi}_z$$

where $\widetilde{\xi}_z$ is the vector field $z' \mapsto z' - z$ and $i_\Gamma\xi_z$ the *index of the loop* Γ *around* z. To construct the loop Γ, one will construct two paths γ and γ' such that $i_\gamma\widetilde{\xi} = i_{\gamma'}\widetilde{\xi} \neq 0$, the first one that joins $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$, the second one that joins $\mathbf{R} \times \{1\}$ to $\mathbf{R} \times \{0\}$. Indeed the loop $\Gamma = \gamma\delta\gamma'\delta'$ obtained by adding horizontal segments on each boundary line will satisfy

$$i_\Gamma\widetilde{\xi} = i_\gamma\widetilde{\xi} + i_\delta\widetilde{\xi} + i_{\gamma'}\widetilde{\xi} + i_{\delta'}\widetilde{\xi} = 2i_\gamma\widetilde{\xi} \neq 0,$$

because the vector field $\widetilde{\xi}$ stays horizontal on the boundary lines, which implies that $i_\delta\widetilde{\xi} = i_{\delta'}\widetilde{\xi} = 0$.

The idea of Birkhoff in [Bi1] to construct these paths was to compose F with a small positive vertical translation H_ε. The iterates of $\mathbf{T}^1 \times \{0\}$ by the perturbed map $F_\varepsilon = H_\varepsilon \circ F$ are pairwise disjoint, and they are not all included in the annulus because F_ε also preserves the area. Consider the lift $f_\varepsilon = h_\varepsilon \circ f$ of F_ε that is close to f and the vector field $\xi_\varepsilon\ : z \mapsto f_\varepsilon(z) - z$. One can choose an arc that joins a point $z \in \mathbf{T}^1 \times \{0\}$ to $F_\varepsilon(z)$ whose iterates are not all included in $\mathbf{A}$. By concatenating all the iterates of α, one gets a simple arc which contains a sub-arc that joins $\mathbf{T}^1 \times \{0\}$ to $\mathbf{T}^1 \times \{1\}$ and that can be lifted to an arc γ in $\widetilde{\mathbf{A}}$. The variation of angle of ξ_ε on γ may be computed, one finds $i_\gamma \xi_\varepsilon = -\dfrac{1}{2}$. As seen above, that implies that f_ε has at least two fixed points whose T orbits are distinct and, by a limit argument, that f has at least one fixed point. Contrarily to what Birkhoff asserted, it does not necessarily imply that f has two fixed points whose T orbits are distinct because a pair of fixed points of f_ε could coincide after passing to the limit. This minor error was corrected by Birkhoff in [Bi3]. A very simple way to correct it to say the following; If $z_0 = (x_0, y_0)$ is an isolated point of $\mathrm{Fix}_*(F)$, one can find a continuous real function ψ on $\mathbf{T}^1$ that vanishes in a neighborhood of x_0 and take positive values outside. All the arguments above are still valid if one replaces the formula $H_\varepsilon(x, y) = (x, y + \varepsilon)$ by $H_\varepsilon(x, y) = (x, y + \varepsilon\psi(x))$ but in that case every set $\mathrm{Fix}_*(F_\varepsilon)$ contains a point outside a fixed neighborhood of z_0 and the limit argument will give us another fixed point.

There are two annoying points in the arguments above. First, one needs to make a perturbation in a larger annulus than $\mathbf{A}$, which seems a little bit artificial, secondly one strongly uses the fact that the perturbation H_ε preserves the Lebesgue measure. It would not be so easy to generalize the previous arguments to the case where F preserves a general finite measure with total support. We will explain now how to construct our paths γ and γ' in this more general situation (see [LeW]). In fact we will do it in the more general case where F has no wandering point. Let us begin by a simple remark. Suppose that γ is a vertical segment in $\widetilde{\mathbf{A}}$ oriented upwards that joins $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$. One sees easily that $i_\gamma \widetilde{\xi} = -1/2$ in the case where the vector fields $\widetilde{\xi}$ never points vertically backwards on γ. This means that there is no point on γ whose image belongs to γ but below. We will see that this computation works for many others paths. Let G be a homeomorphism of a metric space X. A *positive path* of G is a path $\gamma : I \to X$ defined on an interval I such that for every t, t' in

I, one has:

$$t' \geq t \Rightarrow G(\gamma(t')) \neq \gamma(t).$$

Observe that a positive path γ does not meet the fixed point set, that any sub-path of γ is positive and that the images $G^k \circ \gamma$, $k \in \mathbf{Z}$, are also positive.

PROPOSITION 9 : *Let f be a homeomorphism of $\widetilde{\mathbf{A}}$ that satisfies the boundary twist condition (which means the condition* **i)** *of Theorem 1). If γ is a positive path of f that joins a boundary line of $\widetilde{\mathbf{A}}$ to the other one, then $i_\gamma \xi = -\frac{1}{2}$, where $\xi(z) = f(z) - z$.*

Proof. We write the proof in the case where γ joins $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$, the other case being similar. The boundary of the symplex

$$\Delta = \{(t, t') \in I^2 \,|\, t' \geq t\}$$

may be written $\partial\Delta = \delta_d \delta_h \delta_v$ where δ_d is the diagonal, δ_h the horizontal segment and δ_v the vertical one. The path γ beings positive, the map

$$\Phi \ : (t, t') \mapsto f(\gamma(t')) - \gamma(t)$$

does not vanish on Δ and one has

$$\int_{\Phi \circ \delta_d} d\theta + \int_{\Phi \circ (\delta_h \delta_v)} d\theta = \int_{\Phi \circ \partial\Delta} d\theta = \int_{\Phi \circ \Delta} d\, d\theta = 0.$$

Observe now that the image by Φ of each segment δ_h and δ_v does not intersect the vertical half-line $\{0\} \times (-\infty, 0]$. This implies that

$$i_\gamma f = \int_{\Phi \circ \delta_d} d\theta = -\int_{\Phi \circ (\delta_h \delta_v)} d\theta = -\frac{1}{2}.$$

□

It remains to prove the existence of such paths. We will use the general following result of topological dynamics

LEMMA 10 : *Suppose that X is a connected and locally path-connected metric space and that H is a fixed point free homeomorphism of X with no wandering point. If $Z \subset X$ satisfies $H(Z) \subset Z$, then for every $z \in X$ there exists a positive path of H that joins Z to z.*

Proof. One must prove the set Y of points that may be joined by a positive path of H whose origin belongs to Z coincides with X. The space X being connected, it is sufficient to prove that $\overline{Y} \subset \mathrm{Int}(Y)$. Fix $z_0 \in \overline{Y}$. By hypothesis, one can find a path-connected neighborhood V of z_0 such that $\overline{V} \cap H(\overline{V}) = \emptyset$. We will prove that $V \subset Y$.

The fact that $z_0 \in \overline{Y}$ implies that there exists a positive path $\gamma_0 : [0,1] \to X$ from Z to V. The closures of the subsets $J = \gamma_0^{-1}(V)$ and $J' = \gamma_0^{-1}(H(V))$ do not intersect because $\overline{V} \cap H(\overline{V}) = \emptyset$. This implies that $\inf J \neq \inf J'$ (beware that J' can be empty).

Suppose first that $\inf J < \inf J'$ (this includes the case where $J' = \emptyset$). In that case, there is a sub-path γ_1 of γ_0 from Z to V that does not meet $H(V)$. For every $z \in V$ one can find a path γ inside V that joins the final end z_1 of γ_1 to z. The path $\gamma_2 = \gamma_1\gamma$ is positive because γ_1 is positive and $H(\gamma)$ is disjoint both from γ and γ_1. This implies that $z \in Y$.

Suppose now that $\inf J' < \inf J$. In that case, there is a sub-path γ_1 of γ_0 from Z to $H(V)$ that does not meet V. We denote by z_1 its final end. The point $H(z_1)$ does not belong to γ_1 because this path is positive. The path being compact, one can find a path-connected neighborhood $U \subset H(V)$ of z_1 such that $H(U)$ does not intersect γ_1. The set U being non wandering, one can find a point $z_2 \in U$ whose positive orbit meets $H^{-1}(U) \subset V$. Choose a path γ inside U that joins z_1 to z_2. The path $\gamma_2 = \gamma_1\gamma$ does not meet V and is positive because γ_1 is positive and $H(\gamma)$ is disjoint from γ_1 and γ. Let us consider the first integer $k \geq 1$ such that $H^k(\gamma_2) \cap V \neq \emptyset$. Since $H(Z) \subset Z$, the path $H^k(\gamma_2)$ is a positive path from Z to V that does not meet $H(V)$. We conclude like in the first case. $\square$

Let us explain now how to construct our positive paths to prove Poincaré-Birkhoff Theorem. The map $f - \mathrm{Id}$ being bounded on $\widetilde{\mathbf{A}}$, there exists an integer N such that the rotation number of every fixed point of F is smaller than N. This implies that the homeomorphism F' of the annulus $\mathbf{A}' = \mathbf{R}/N\mathbf{Z} \times [0,1]$ lifted by f has no fixed points but the ones that are lifted to fixed points of f. As one supposes that $\mathrm{Fix}_*(F)$ is finite, one deduces that $\mathrm{Fix}(F')$ is finite and that $\mathbf{A}'\setminus \mathrm{Fix}(F')$ is connected. Moreover F' has no wandering point because it preserves the area. Applying the lemma to $X = \mathbf{A}' \setminus \mathrm{Fix}(F')$, to $H = F'|_{\mathbf{A}'\setminus\mathrm{Fix}(F')}$ and to $Z = \mathbf{R}/N\mathbf{Z} \times \{0\}$ or $Z = \mathbf{R}/N\mathbf{Z} \times \{1\}$, one constructs a positive path of F' from one of the boundary circle of $\mathbf{A}'$ to the other one. Such a path is lifted to a positive path of f from the corresponding boundary line to the other one.

Remark. The proof above clearly works if F preserves any finite measure with total support. Let us explain why it works in the case where F has no wandering point. It is sufficient to prove that F' has no wandering point. Let T' be a generator of the (finite) group of automorphisms of the covering space $\mathbf{A}'$. The fact that F has no wandering point implies that for every non empty open set $U \subset \mathbf{A}'$, there exists $q \geq 1$ and $p \in \mathbf{Z}$ such that $F'^q(U) \cap T'^p(U) \neq \emptyset$. Let us fix a non empty open set $U_0 \subset \mathbf{A}'$ and define a sequence $(U_k)_{k\geq 0}$ of non empty open sets where U_{k+1} may be written $U_{k+1} = F'^{q_k}(U_k) \cap T'^{p_k}(U_k)$. One deduces that for every $k' > k$, one has $U_{k'} \subset F'^{q_k+\cdots q_{k'-1}}(U_k) \cap T'^{p_k+\cdots p_{k'-1}}(U_k)$. One can find $k' > k$ such that $p_k + \cdots + p_{k'-1} = 0 \mod n$. This implies that U_k is non-wandering and therefore that U_0 itself is non-wandering.

4.3. **Franks' ideas.**

In [Fr1], Franks gave an alternative proof of Poincaré-Birkhoff theorem, valid also in the case where F has no wandering point and that uses the following result of Brouwer [Brou] (see also Fathi [Fa]) :

Theorem 11 : *Let g be an orientation preserving homeomorphism of $\mathbf{R}^2$ having a periodic point of period $q \geq 2$. Then there exists a loop Γ such that $i_\Gamma Y = 1$, where $Y(x) = g(x) - x$.*

In fact, Franks observed that such a loop can be constructed under the weaker hypothesis of existence of a *periodic free disk chain.* It is a family $(U_r)_{r\in\mathbf{Z}/n\mathbf{Z}}$ of pairwise disjoint free (i.e. disjoint from their image) topological open disks such that for every $r \in \mathbf{Z}/n\mathbf{Z}$, one of the positive iterates of U_r meets U_{r+1}. Let us explain why. One can choose in each disk U_r a point z_r and an integer $m_r \geq 1$ such that $g^{m_r}(z_r) \in U_{r+1}$. One can always suppose that the points $g^i(z_r)$, $0 < r < m_r$, $r \in \mathbf{Z}/n\mathbf{Z}$, lie outside $U = \bigcup_{r\in\mathbf{Z}/n\mathbf{Z}} U_r$ and are distinct. One can construct an orientation preserving homeomorphism h of $\mathbf{R}^2$, supported on U, that sends $g^{m_r}(z_r)$ on z_{r+1}, and then an isotopy $(h_t)_{t\in[0,1]}$ from identity to h, supported on U. Observe now that $h\circ g$ has a periodic orbit of periodic $\sum_{r\in\mathbf{Z}/n\mathbf{Z}} m_r$, which implies that there exists a loop Γ such that $i_\Gamma Y_1 = 1$, where $Y_t(x) = h_t \circ g(x) - x$. The disks being free, the fixed points of each map $h_t \circ g$ coincide with the fixed points of g. This implies that the map $t \mapsto i_\Gamma Y_t$ is well defined on $[0,1]$. As it is continuous, takes only integer values and is equal to 1 at $t = 1$, it is also equal to 1 at $t = 0$.

Going back to Poincaré-Birkhoff Theorem, the facts that F has no wandering point and that f satisfies the boundary twist condition permitted him to construct such a chain under the condition of finiteness of $\mathrm{Fix}_*(F)$, which implies that this set had at least two elements. The proof of Franks is simple but based on Brouwer's theory which is not so easy (Kérekjártó already observed the link between Brouwer's theory and Poincaré-Birkhoff Theorem [K]). The big interest of Franks' method is that it permits to prove natural generalizations of Poincaré-Birkhoff Theorem, mainly by weakening the twist condition, which turned out to be useful in the study of homeomorphisms of surfaces. Let us give here an example of such a generalization of Poincaré-Birkhoff Theorem:

THEOREM 12 : *Let F be a homeomorphism of $\mathbf{A}$ isotopic to the identity and f a lift to $\widetilde{\mathbf{A}}$. We suppose that:*

i) *F admits two periodic points of distinct rotation number $\rho_0 \in \mathbf{Q}$ and $\rho_1 \in \mathbf{Q}$.;*

ii) *F has no wandering point.*

Then, every rational number $\rho = p/q$ (written in an irreducible way) in the open interval bounded by ρ_0 and ρ_1, is the rotation number of a periodic orbit of F of period q.

5. Poincaré-Birkhoff Theorem and symplectic geometry

In the previous section, the Poincaré-Birkhoff theorem has been viewed from a topological angle. We will see here another interpretation of this theorem in the case of diffeomorphisms.

Let I be one of the intervals $[0,1]$, $[0,+\infty)$ or $\mathbf{R}$. Let F be a C^1 diffeomorphism of the annulus $\mathbf{T}^1 \times I$, isotopic to the identity and preserving the area and f a lift of F to $\mathbf{R} \times I$. Let us consider the two coordinates x and y as functions on $\mathbf{R} \times I$, as the coordinates $x' = p_1 \circ f(x,y)$ and $y' = p_2 \circ f(x,y)$ of the image. The fact that f preserves the area means that the 2-form $dx \wedge dy$ is invariant by f or equivalently that the 1-form $f^*(ydx) - ydx = y'dx' - ydx$ is closed. The space $\mathbf{R}^2$ being simply connected, this implies that the form $y'dx' - ydx$ is exact: there exists a C^2-function $h\ : \mathbf{R}^2 \to \mathbf{R}$ such that $dh = y'dx' - ydx$. The form $y'dx' - ydx$ being invariant by T lifts a 1-form on $\mathbf{T}^1 \times I$ that may be written in a similar way $F^*(ydx) - ydx = y'dx' - ydx$, each term having a sense in $\mathbf{T}^1 \times I$. Moreover, one deduces that the function $h \circ T - h$ is constant. The case where $h \circ T = h$ corresponds to the case where h lifts a function $H\ : \mathbf{T}^1 \times I \to \mathbf{R}$, or equivalently to the case where the form $F^*(ydx) - ydx = y'dx' - ydx$ is exact on $\mathbf{T}^1 \times I$. In that case, one says that F is *exact symplectic.* If $\Gamma \subset \mathbf{T}^1 \times I$ is an essential simple loop, the value of $\int_\Gamma y'dx' - ydx$ does not depend on Γ (up to the sign which depends of the orientation) and measures the "algebraic area" between Γ and $F(\Gamma)$. The diffeomorphism is exact-symplectic if and only if this constant is equal to zero. In particular an exact symplectic diffeomorphism satisfies the intersection assumption **ii")** decribed in the first section. Observe that the existence of an invariant essential loop implies that F is exact symplectic. In particular, F is always exact symplectic in the case where $I = [0,1]$ or $I = [0,+\infty)$.

As explained above, every area preserving diffeomorphism on $\mathbf{A}$ that is isotopic to the identity is exact symplectic. Equivalently it is the time one map of an *Hamiltonian isotopy* $(F_t)_{t\in\mathbf{R}}$: there exists a C^2 function $H\ : \mathbf{A} \times \mathbf{R} \to \mathbf{R}$, periodic in time of period 1, such that for every point $(x, y) \in \mathbf{A}$, the function $t \mapsto F_t(x, y)$ is the solution of the Hamiltonian system

$$\begin{cases} x' = \dfrac{\partial H}{\partial y} \\[2em] y' = -\dfrac{\partial H}{\partial x} \end{cases}$$

Moreover if f is a given lift of F, one can suppose that $f = f_1$ where $(f_t)_{t \in \mathbf{R}}$ is the lifted isotopy such that f_0 is the identity.

A very special class of area preserving diffeomorphisms on $\mathbf{A}$ is the class of time one maps of time independent Hamiltonian isotopies (the case where the function H above does not depend on t). In this case, each function $t \mapsto F_t(x, y)$ describes the orbit of a time independent vector field. Such orbits are easy to describe. Indeed, H is an integral of motion, it is constant on each orbit. This implies that the orbit of a critical point of H is reduced to this point, which implies that each critical point is lifted to fixed points of f, and that the orbit of a non critical point is the connected component of a level curve in the complement of the critical points. Let us explain why the Poincaré-Birkhoff Theorem is easy to prove in this situation. Indeed if F satisfies the assumptions of this theorem, the boundary twist condition implies that the gradient vector field $\nabla H(x, y) = \left(\dfrac{\partial H}{\partial x}, \dfrac{\partial H}{\partial y}\right)$ points outwards. In particular the minimum of H is not on the boundary of $\mathbf{A}$ and therefore corresponds to a critical point. For topological reasons, there is another critical point, which is a minimax of H. Otherwise every point of $\mathbf{A}$ will be attracted to the minimum of H under the dynamics of $-\nabla H$, which is impossible.

There is another case, already observed by Poincaré, where the Poincaré-Birkhoff Theorem can be proven easily, the case where F is "close to the identity". We keep the notations above with an area preserving diffeomorphim F of $\mathbf{A}$ isotopic to the identity. The 1-form $(x - x')dy + (y' - y)dx'$ defined on $\mathbf{R} \times [0, 1]$ is closed and invariant by T, so it lifts a closed 1-form on $\mathbf{A}$ (the function $x' - x$

is well defined on $\mathbf{A}$). The form vanishes, when integrated on the boundary curves. This implies that it is exact and may be written $(x - x')dy + (y' - y)dx' = dH$, where H is a C^2-function defined on $\mathbf{A}$ which is constant on each boundary line. In the case where the map $(x, y) \mapsto (x', y)$ is C^1 diffeomorphism of $\widetilde{\mathbf{A}}$, which means that $\partial(p_1 \circ f)(x, y)/\partial x > 0$, the couple (x', y) defines a system of coordinates on $\mathbf{R}\times[0, 1]$. In this case, any critical point of H is lifted to fixed points of f. Here again, the gradient $\nabla H(x', y) = \left(\dfrac{\partial H}{\partial x'}, \dfrac{\partial H}{\partial y}\right)$ points outwards. Like in the case above, one deduces that it has at least two critical points. One says that H is a *generating function* of F

The previous remarks leaded Arnold to state his famous conjecture for Hamiltonian diffeomorphisms [Ar1]. Let (M, ω) be a symplectic compact manifold. That means that ω is a non degenerate closed 2-form on the (even dimensional) manifold M. If H is a smooth function on M, one can define the induced *Hamiltonian vector field* X_H, which is the symplectic gradient of H: for every $z \in M$ and every $Y \in T_zM$, one has

$$\omega_z(X_H(z), Y) = T_zH(Y).$$

Every critical point of H is a singularity of X_H. So one may minimize the number of singularities of X_H by an integer n_M, equal to the minimum number of critical points that any smooth function defined on M must have. One can define similarly an integer $n'_M \geq n_M$ if we restrict ourselves to the Morse functions, that means functions whose critical points z are all nondegenerate: the bilinear form T_z^2H is non degenerate. The integer n_M is at least equal to the number of charts necessary to cover M and the integer n'_M is at least equal to the sum of the Betti numbers. For example :

$$\begin{cases} n_M = n'_M = 2 & \text{if } M = S^n \text{ is a sphere ,} \\ n_M = 3,\ n'_M = 2 + 2g & \text{if M is a compact oriented surface of genus } g \geq 1, \\ n_M = n + 1,\ n'_M = 2^n & \text{if } M = \mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n \text{ is a torus.} \end{cases}$$

Consider now a smooth function $H : M\times\mathbf{R}\to\mathbf{R}$ periodic in time, of period 1. Define the time dependent vector field X_H, where $X_H(t,.)$ is the symplectic gradient of $H(t,.)$, and the time-one map $F=F_1$ of the identity isotopy $(F_t)_{t\in\mathbf{R}}$ naturally defined by X_H. Say that a fixed point is *contractible* if the loop $(F_t(z))_{t\in[0,1]}$ is homotopic to zero.

Let us state the Arnold's conjecture:

Is the number of contractible fixed points of F minimized by n_M ? Is it minimized by n'_M if every fixed point of F is nondegenerate ?

As we have seen above in the case of the annulus, if F is "close to the identity", one can construct a generating function whose critical points correspond to the contractible fixed points and the conjecture is true. The first proven case of the general conjecture was done by Conley and Zehnder [CoZ] for $\mathbf{T}^{2n}$, the case of the surfaces was done a little bit later independently by Floer [Fl1], and Sikorav [Si]. For the second question, a breakthrough was done by Floer [Fl2] who minimized the number of contractible fixed points by the sum of the Betti numbers under a special "monotone" condition on the symplectic manifold. Floer's result was generalized by Hofer and Salamon [HoSa] and by Ono [O] to the "weakly monotone" case. Now this minoration is known to be true for any symplectic manifold (Liu and Tian [LiT], Fukaya and Ono [FuO]).

6. Appendix: rotation number of homeomorphisms of the circle.

We denote by $\mathrm{Homeo}_+(\mathbf{T}^1)$ the group of orientation preserving homeomorphisms of $\mathbf{T}^1$ and by $\widetilde{\mathrm{Homeo}}_+(\mathbf{T}^1)$ the group of lifts of elements of $\mathrm{Homeo}_+(\mathbf{T}^1)$ to the universal covering space, that means the set of increasing homeomorphims of $\mathbf{R}$ such that $g(x+1)=g(x)+1$, for every $x\in\mathbf{R}$. This last group contains the translations

$\tau_a : x \mapsto x + a$. If one munishes $\widetilde{\mathrm{Homeo}}_+(\mathbf{T}^1)$ with the topology induced by the following distance

$$d(g_0, g_1) = \max\left(\max_{x\in\mathbf{R}} |g_0(x) - g_1(x)|,\ \max_{x\in\mathbf{R}} |g_0^{-1}(x) - g_1^{-1}(x)|\right),$$

one gets a topological group. The continuity of the map $g \mapsto g^{-1}$ is immediate. Let us prove that the map $(g_0, g_1) \mapsto g_0 \circ g_1$ is also continuous. We suppose that

$$\lim_{n\to+\infty} g_{n,0} = g_0, \quad \lim_{n\to+\infty} g_{n,1} = g_1$$

and we want to prove that

$$\lim_{n\to+\infty} g_{n,0} \circ g_{n,1} = g_0 \circ g_1.$$

Fix $\varepsilon > 0$. There exists $N \geq 0$ such that $d(g_{n,0}, g) \leq \varepsilon/2$ for every $n \geq N$. The homeomorphism g_0 being uniformly continuous (it is equal to the sum of the identity and of a continuous periodic map), there exists $\eta > 0$ such that

$$|x - y| \leq \eta \Rightarrow |g_0(x) - g_0(y)| \leq \varepsilon/2.$$

There exists $N' \geq 0$ such that $d(g_{n,1}, g_1) \leq \eta$ for every $n \geq N'$. Therefore, for every $n \geq \max(N, N')$ one has

$$\begin{aligned} |g_{n,0} \circ g_{n,1}(x) - g_0 \circ g_1(x)| &\leq |g_{n,0} \circ g_{n,1}(x) - g_0 \circ g_{n,1}(x)| \\ &\quad + |g_0 \circ g_{n,1}(x) - g_0 \circ g_1(x)| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

With the same proof we show that

$$|g_{n,1}^{-1} \circ g_{n,0}^{-1}(x) - g_1^{-1} \circ g_0^{-1}(x)| \leq \varepsilon,$$

if n is large enough.

Theorem 13: *For every $g \in \widetilde{\mathrm{Homeo}}_+(\mathbf{T}^1)$, there exists a real number ρ such that for every $x \in \mathbf{R}$ and every $k \in \mathbf{Z}$, one has*

$$-1 < g^k(x) - x - k\rho < 1.$$

In particular, ρ satisfies

$$\rho = \lim_{k \to \pm\infty} \frac{g^k(x)}{k},$$

it is unique, called the rotation number *of g and denoted by $\rho(g)$.*

Proof. We write $g(x) = x + \psi(x)$, where $\psi : \mathbf{R} \to \mathbf{R}$ is 1-periodic.

LEMMA 14: *For every real numbers x, y, one has $-1 < \psi(y) - \psi(x) < 1$.*

Proof. Denote by y' the unique element in $[x, x+1)$ such that $y' - y \in \mathbf{Z}$. From the inequalities

$$x + \psi(x) \leq y' + \psi(y') < x + 1 + \psi(x)$$

one deduces

$$x - x - 1 < x - y' \leq \psi(y') - \psi(x) < x + 1 - y' \leq x + 1 - x$$

and

$$-1 < \psi(y) - \psi(x) < 1.$$

□

Applying the previous lemma to each iterate g^k, $k \geq 1$, one gets

$$0 \leq M_k - m_k < 1,$$

where

$$m_k = \min_{x \in \mathbf{R}} g^k(x) - x \text{ and } M_k = \max_{x \in \mathbf{R}} g^k(x) - x.$$

If k and k' are two positive integers and $x \in \mathbf{R}$, one knows that

$$g^{k+k'}(x) - x = g^k(g^{k'}(x)) - g^{k'}(x) + g^{k'}(x) - x \leq M_k + M_{k'}$$

and similary that

$$m_k + m_{k'} \leq g^{k+k'}(x) - x.$$

This implies

$$m_k + m_{k'} \leq m_{k+k'} \leq M_{k+k'} \leq M_k + M_{k'}.$$

Using a simple induction argument, one deduces that for every $k \geq 1$ and $k' \geq 1$, one has $M_{k'k} \leq k'M_k$ and $m_{k'k} \geq km_{k'}$ which implies that
$$\frac{m_{k'}}{k'} \leq \frac{m_{k'k}}{k'k} \leq \frac{M_{k'k}}{k'k} \leq \frac{M_k}{k}.$$
The set of numbers $\dfrac{m_k}{k}$ is located on the left of the set of numbers $\dfrac{M_k}{k}$. As we know that $\dfrac{M_k}{k} - \dfrac{m_k}{k} < \dfrac{1}{k}$, we deduce that
$$\sup_{k\geq 1} \frac{m_k}{k} = \inf_{k\geq 1} \frac{M_k}{k}.$$

We denote by ρ this commond bound. For every $k \geq 1$, one has $m_k \leq k\rho \leq M_k$. Recall that there exist x_k and y_k such that $g^k(x_k)-x_k = m_k$ and $g^k(y_k) - y_k = M_k$. By the Intermediate Value Theorem, this implies that there exists z_k such that $g^k(z_k) - z_k = k\rho$. Applying Lemma 14 to the fonction g^k and the points x and z_k, one obtains
$$-1 < g^k(x) - x - k\rho < 1.$$

The theorem has been proved if k is positive. As it is clearly obvious when $k = 0$, it remains to study the case where k is negative. If one applies the theorem to the point $g^k(x)$ for the iterate g^{-k}, one gets
$$-1 < g^{-k}(g^k(x)) - g^k(x) + k\rho < 1,$$
which implies
$$-1 < g^k(x) - x - k\rho < 1.$$

□

Remarks Observe that in the proof of the theorem, the following has been stated: if $p \in \mathbf{Z}$ and $q \geq 1$ are two integers, one has $\rho(g) = p/q$ if and only if there exists $x \in \mathbf{R}$ such that $g^q(x) = x + p$; one has $\rho(g) > p/q$ if and only if $g^q(x) > x + p$ for every $x \in \mathbf{R}$; one has $\rho(g) < p/q$ if and only if $g^q(x) < x + p$ for every $x \in \mathbf{R}$.

Proposition: *The following properties holds:*

i) *the rotation number of τ_a is a;*

ii) *for every* $g \in \widetilde{\mathrm{Homeo}}_+(\mathbf{T}^1)$ *and every* $p \in \mathbf{Z}$, *one has* $\rho(g \circ \tau_p) = \rho(g) + p$;

iii) *for every* $g \in \widetilde{\mathrm{Homeo}}_+(\mathbf{T}^1)$ *and every* $q \in \mathbf{Z}$, *one has* $\rho(g^q) = q\rho(g)$;

iv) *if* $g_0 \leq g_1$, *then* $\rho(g_0) \leq \rho(g_1)$;

v) *the map* $g \mapsto \rho(g)$ *is continuous if* $\widetilde{\mathrm{Homeo}}_+(\mathbf{T}^1)$ *is endowed with the topology defined at the beginning of the appendix.*

Proof. To get **i)**, write

$$\rho(\tau_a) = \lim_{k\to+\infty} \frac{\tau_a^k(x)}{k} = \lim_{k\to+\infty} \frac{x+ka}{k} = a.$$

To get **ii)**, observe that

$$(g \circ \tau_p)^k = g^k \circ \tau_p^k$$

because g and τ_p commute. Thus

$$\rho(g \circ \tau_p) = \lim_{k\to+\infty} \frac{(g \circ \tau_p)^k(x)}{k} = \lim_{k\to+\infty} \frac{g^k(x)+kp}{k} = \rho(g) + p.$$

To get **iii)**, write

$$\rho(g^q) = \lim_{k\to+\infty} \frac{(g^q)^k(x)}{k} = \lim_{k\to+\infty} \frac{g^{qk}(x)}{k} = q\rho(g).$$

To get **iv)**, one must prove by induction on $k \geq 1$ that $g_0^k \leq g_1^k$. The assertion is supposed to be true for $k = 1$. If it is true for $k-1$, note that for every $x \in \mathbf{R}$, one has

$$g_0^{k+1}(x) = g_0(g_0^k(x)) \leq g_0(g_1^k(x)) \leq g_1(g_1^k(x)) = g_1^{k+1}(x).$$

Therefore

$$\rho(g_0) = \lim_{k\to+\infty} \frac{g_0^k(x)}{k} \leq \lim_{k\to+\infty} \frac{g_1^k(x)}{k} = \rho(g_1).$$

To get **v)**, observe first that the maps $g \mapsto g^k$ are continuous, because it the case for the map $(g_0, g_1) \to g_0 \circ g_1$. Suppose that $\lim_{n\to+\infty} g_n = g$. Fix $\varepsilon > 0$, then choose $k \geq 1$ such that $3 < k\varepsilon$. Fix $x \in \mathbf{R}$. There exists $N \geq 1$ such that $-1 < g^k(x) - g_n^k(x) < 1$, for every $n \geq N$. From the inequalities

$$
\begin{aligned}
-1 &< g_n^k(x) - x - k\rho(g_n) < 1, \\
-1 &< -g^k(x) + x + k\rho(g) < 1, \\
-1 &< g^k(x) - g_n^k(x) < 1,
\end{aligned}
$$

one deduces that

$$-3 < k(\rho(g) - \rho(g_n)) < 3,$$

which implies

$$-\varepsilon < \rho(g) - \rho(g_n) < \varepsilon.$$

□

Remarks

• Note that $\rho(g^{-1}) = -\rho(g)$ and that $\rho(g^q \circ T^p) = q\rho(g) + p$, if p and q are integers.

• The assertion **ii)** implies that the class $\rho(g) + \mathbf{Z} \in \mathbf{T}^1$ does not depend on the lift g of a given homeomorphism $G \in \mathrm{Homeo}_+(\mathbf{T}^1)$. It depends only on G, it is the *rotation number* $\rho(G) \in \mathbf{T}^1$ of G.

• In general one does not have $\rho(g_0 \circ g_1) = \rho(g_0) + \rho(g_1)$. However this formula is true if g_0 and g_1 commute. Indeed, from

$$
\begin{aligned}
-1 &< g_0^k \circ g_1^k(x) - g_1^k(x) - k\rho(g_0) < 1, \\
-1 &< g_1^k(x) - x - k\rho(g_1) < 1,
\end{aligned}
$$

one gets

$$-2 < (g_0 \circ g_1)^k(x) - x - k(\rho(g_0) + \rho(g_1)) < 2,$$

which implies

$$\rho(g_0 \circ g_1) = \lim_{k \to +\infty} \frac{(g_0 \circ g_1)^k(x)}{k} = \rho(g_0) + \rho(g_1).$$

Example Fix $\alpha \in (-\frac{1}{2\pi}, \frac{1}{2\pi})$ and define for every $t \in \mathbf{R}$ the real map

$$g_t \ : x \mapsto x + \alpha \sin(2\pi x) + t.$$

One can verify that $g_t \in \widetilde{\mathrm{Homeo}}_+(\mathbf{T}^1)$ and that $t \mapsto g_t$ is continuous. This implies by the previous proposition that the map $r\ : t \mapsto \rho(g_t)$ is continuous, non decreasing and satisfies $r(t+1) = r(t) + 1$, for every $t \in \mathbf{R}$. One may prove that each set $r^{-1}(\{a\})$ is a non trivial closed interval if $a \in \mathbf{Q}$ and is reduced to a point if $a \notin \mathbf{Q}$.

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SYSTEMS OF POLYNOMIAL EQUATIONS

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Lecture 1. Counting complex roots

In this lecture[1], we will mostly look at equations over the field of complex numbers. The case of real equations will be dealt in the last lecture.

Finding or even counting the solutions of specific systems of polynomials is *hard* in the complexity theory sense. Therefore, instead of looking at particular equations, we consider linear spaces of equations. Several bounds for the number of roots are known to be true *generically*. As many definitions of genericity are in use, we should be more specific.

Definition 1.1 (Zariski topology)**.** A set $V \subseteq \mathbb{C}^N$ is *Zariski closed* if and only if it is of the form

$$V = \{\mathbf{x} : f_1(x) = \cdots = f_s(x) = 0\}$$

for some finite (possibly empty) collection of polynomials $f_1, \ldots, f_s$. A set is *Zariski open* if it is the complementary of a Zariski closed set.

In particular, the empty set and the total space $\mathbb{C}^N$ are simultaneously open and closed.

Definition 1.2. We say that a property holds for a *generic* $\mathbf{y} \in \mathbb{C}^N$ (or more loosely for a generic choice of $y_1, \ldots, y_N$) when the set of $\mathbf{y}$ where this property holds contains a non-empty Zariski open set.

[1]This is an edited version of Chapter 1 in Gregorio Malajovich, *Nonlinear Equations*, Publicações Matemáticas do IMPA – 28º Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2011.

A property holding generically will also hold almost everywhere (in the measure-theory sense).

Exercise 1.1. Show that a finite union of Zariski closed sets is Zariski closed.

The proof that an arbitrary intersection of Zariski closed sets is Zariski closed (and hence the Zariski topology is a topology) is in [9, Cor.2.7].

1. Bézout's theorem

Below is *the* classical theorem about root counting. The notation $\mathbf{x}^{\mathbf{a}}$ stands for

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

The *degree* of a multi-index $\mathbf{a}$ is $|\mathbf{a}| = a_1 + a_2 + \cdots + a_n$.

Theorem 1.3 (Étienne Bézout, 1730–1783)**.** *Let $n, d_1, \ldots, d_n \in \mathbb{N}$. For a generic choice of the coefficients $f_{i\mathbf{a}} \in \mathbb{C}$, the system of equations*

$$\begin{aligned} f_1(\mathbf{x}) &= \sum_{|\mathbf{a}| \le d_1} f_{1\mathbf{a}} \mathbf{x}^{\mathbf{a}} \\ &\vdots \\ f_n(\mathbf{x}) &= \sum_{|\mathbf{a}| \le d_n} f_{n\mathbf{a}} \mathbf{x}^{\mathbf{a}} \end{aligned}$$

has exactly $B = d_1 d_2 \ldots d_n$ roots $\mathbf{x}$ in $\mathbb{C}^n$. The number of isolated roots is never more than B.

This can be restated in terms of homogeneous polynomials with roots in projective space $\mathbb{P}^n$. We introduce a new variable x_0 (the *homogenizing variable*) so that all monomials in the i-th equation have the same degree. We denote by f_i^{h} the homogenization of f_i,

$$f_i^{\mathrm{h}}(x_0, \ldots, x_n) = x_0^{d_i} f_i\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

Once this is done, if $(x_0, \cdots, x_n)$ is a simultaneous root of all f_i^{h}'s, so is $(\lambda x_0, \cdots, \lambda x_n)$ for all $\lambda \in \mathbb{C}$. Therefore, we count complex 'lines' through the origin instead of points in $\mathbb{C}^{n+1}$.

The space of complex lines through the origin is known as the projective space $\mathbb{P}^n$. More formally, $\mathbb{P}^n$ is the quotient of $(\mathbb{C}^{n+1})_{\neq 0}$ by the multiplicative group $\mathbb{C}_\times$.

A root $(z_1, \ldots, z_n) \in \mathbb{C}^n$ of $\mathbf{f}$ corresponds to the line $(\lambda, \lambda z_1, \ldots, \lambda z_n)$, also denoted by $(1 : z_1 : \cdots : z_n)$. That line is a root of $\mathbf{f}^{\mathrm{h}}$.

Roots $(z_0 : \cdots : z_n)$ of $\mathbf{f}^{\mathrm{h}}$ are of two types: if $z_0 \neq 0$, then z corresponds to the root $(z_1/z_0, \ldots, z_n/z_0)$ of $\mathbf{f}$, and is said to be *finite*. Otherwise, z is said to be *at infinity*.

We will give below a short and sketchy proof of Bézout's theorem. It is based on four basic facts, not all of them proved here.

The first fact is that *Zariski open sets are path-connected.* Suppose that V is a Zariski closed set, and that $\mathbf{y}_1 \neq \mathbf{y}_2$ are not points of V. (This already implies $V \neq \mathbb{C}^n$). We claim that there is a path connecting $\mathbf{y}_1$ to $\mathbf{y}_2$ not cutting V. It suffices to exhibit a path in the complex 'line' L passing through $\mathbf{y}_1$ and $\mathbf{y}_2$, which can be parameterized by

$$(1-t)\mathbf{y}_1 + t\mathbf{y}_2, \quad t \in \mathbb{C}.$$

The set $L \cap V$ is the set of the simultaneous zeros of polynomials $f_i((1-t)\mathbf{y}_1 + t\mathbf{y}_2)$, where f_i are the defining polynomials of V. Hence $L \cap V$ is the zero set of the greater common divisor of those polynomials. It is a finite (possibly empty) set of points. Hence there is a path between $\mathbf{y}_1$ and $\mathbf{y}_2$ not crossing those points.

The second fact is a classical result in Elimination Theory. *Given a system of homogeneous polynomials* $\mathbf{g}(\mathbf{x})$ *with indeterminate coefficients, the coefficient values such that there is a common solution in* $\mathbb{P}^n$ *are a Zariski closed set.* This is the Main Theorem of Elimination Theory [9, Th.2.33].

The third fact is that *the set of polynomial systems with a root at infinity is Zariski closed.* A system $\mathbf{g}$ has a root $\mathbf{x}$ at infinity if and only if for each i,

$$G_i(x_1, \dots, x_n) \stackrel{\text{def}}{=} g_i^{\text{h}}(0, x_1, \dots, x_n) = 0$$

for some choice of the $x_1, \dots, x_n$. Now, each G_i is homogeneous of degree d_i in n variables. By the fact #2, this happens only for the G_i (hence the g_i) in some Zariski-closed set.

The fourth fact is that *the number of isolated roots is lower semi-continuous* as a function of the coefficients of the polynomial system $\mathbf{f}$. This is a topological fact about systems of complex analytic equations [9, Cor 2.9]. It is not true for real analytic equations.

Sketch: Proof of Bézout's Theorem. We consider first the polynomial system

$$\begin{aligned} f_1^{\text{ini}}(\mathbf{x}) &= x_1^{d_1} - 1 \\ &\vdots \\ f_n^{\text{ini}}(\mathbf{x}) &= x_n^{d_n} - 1. \end{aligned}$$

This polynomial has exactly $d_1 d_2 \cdots d_n$ roots in $\mathbb{C}^n$ and no root at infinity. The derivative $D\mathbf{f}(z)$ is non-degenerate at any root z.

The derivative of the evaluation function $\text{ev} : \mathbf{f}, \mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ is

$$\dot{\mathbf{f}}, \dot{\mathbf{x}} \mapsto D\mathbf{f}(\mathbf{x})\dot{\mathbf{x}} + \dot{\mathbf{f}}(\mathbf{x}).$$

Assume that $\mathbf{f}_0(\mathbf{x}_0) = 0$ with $D\mathbf{f}_0(\mathbf{x}_0)\dot{\mathbf{x}}$ non-degenerate. Then the derivative of ev with respect to the $\mathbf{x}$ variables is an isomorphism. By the implicit function theorem, there is a neighborhood $U \ni \mathbf{f}_0$ and a function $\mathbf{x}(\mathbf{f}) : U \to \mathbb{C}^n$ so that $\mathbf{f}(\mathbf{x}_0) = \mathbf{f}_0$ and

$$\text{ev}(\mathbf{f}(\mathbf{x}), \mathbf{x}) \equiv 0.$$

Now, let

$$\Sigma = \left\{ \mathbf{f} : \exists \mathbf{x} \in \mathbb{P}^{n+1} : \mathbf{f}^{\text{h}}(1, \mathbf{x}) = 0 \text{ and } \left(\det D\mathbf{f}(\cdot)\right)^{\text{h}}(1, \mathbf{x}) = 0 \right\}.$$

By elimination theory, Σ is a Zariski closed set. It does not contain $\mathbf{f}^{\text{ini}}$, so its complement is not empty.

Let $\mathbf{g}$ be a polynomial system not in Σ and without roots at infinity. (Fact 3 says that this is true for a *generic* $\mathbf{g}$). We claim that $\mathbf{g}$ has the same number of roots as $\mathbf{f}^{\text{ini}}$.

Since Σ and the set of polynomials with roots at infinity are Zariski closed, there is a smooth path (or *homotopy*) between $\mathbf{f}^{\text{ini}}$ and $\mathbf{g}$ avoiding those sets. Along this path, locally, the root count is constant. Indeed, let $I \subseteq [0, 1]$ be the maximal interval so that the implicit function $\mathbf{x}_t$ for $\mathbf{f}_t(\mathbf{x}_t) \equiv 0$ can be defined. Let $t_0 = \sup I$. If $1 \neq t_0 \in I$, then (by the implicit function theorem) the implicit function $\mathbf{x}_t$ can be extended to some interval $(0, t_0 + \epsilon)$ contradicting that $t_0 = \sup I$. So let's suppose that $t_0 \notin I$. The fact that $\mathbf{f}_{t_0}$ has no root at infinity makes $\mathbf{x}_t$ convergent when $t \to t_0 \pm \epsilon$. Hence $\mathbf{x}_t$ can be extended to the closed interval $[0, t_0]$, another contradiction. Therefore $I = [0, 1]$.

Thus, $\mathbf{f}^{\text{ini}}$ and $\mathbf{g}$ have the same number of roots.

Until now we counted roots of systems outside Σ. Suppose that $\mathbf{f} \in \Sigma$ has more roots than the Bézout bound. By lower semi-continuity of the root count, there is a neighborhood of $\mathbf{f}$ (in the usual topology) where there are at least as many roots as in $\mathbf{f}$. However, this neighborhood is not contained in Σ, contradiction.

□

2. Shortcomings of Bézout's Theorem

The example below (which I learned long ago from T.Y. Li) illustrates one of the major shortcomings of Bézout's theorem:

Example 1.4. Let A be a $n \times n$ matrix, and we consider the eigenvalue problem

$$A\mathbf{x} - \lambda\mathbf{x} = 0.$$

Eigenvectors are defined up to a multiplicative constant, so let us fix $x_n = 1$. We have $n-1$ equations of degree 2 and one linear equation. The Bézout bound is $B = 2^{n-1}$.

Of course there should be (generically) n eigenvalues with a corresponding eigenvector. The other solutions given by Bézout bound lie *at infinity*: if one homogenizes the system, say

$$\begin{aligned}
\sum_{j=1}^{n-1} a_{1j}\mu x_j + a_{1n}\mu^2 - \lambda x_1 &= 0 \\
&\vdots \\
\sum_{j=1}^{n-1} a_{n-1,j}\mu x_j + a_{n-1,n}\mu^2 - \lambda x_{n-1} &= 0 \\
\sum_{j=1}^{n-1} a_{nj} x_j + a_{n,n}\mu - \lambda &= 0
\end{aligned}$$

where μ is the homogenizing variable, and then set $\mu = 0$, one gets:

$$\begin{aligned}
-\lambda x_1 &= 0 \\
&\vdots \\
-\lambda x_{n-1} &= 0 \\
\sum_{j=1}^{n-1} a_{nj} x_j - \lambda &= 0
\end{aligned}$$

This defines an $n-2$-dimensional space of solutions at infinity for $\lambda = 0$ and $a_{n1}x_1 + \cdots + a_{n,n-1}x_{n-1} = 0$.

Here is what happened: when $n \geq 2$, no system of the form $A\mathbf{x} - \lambda\mathbf{x} = 0$ can be generic in the space of polynomials systems of degree $(2, 2, \cdots, 2, 1)$. This situation is quite common, and it pays off to refine Bézout's bound.

One can think of the system above as a bi-linear homogeneous system, of degree 1 in the variables $x_1, \ldots, x_{n-1}, x_n$ and degree 1 in variables λ, μ. The equations are now

$$\mu A\mathbf{x} - \lambda\mathbf{x} = 0.$$

The eigenvectors $\mathbf{x}$ are elements of projective space $\mathbb{P}^n$ and the eigenvalue is $(\lambda : \mu) \in \mathbb{P} = \mathbb{P}^1$. Examples of "ghost" roots in $\mathbb{P}^{n+1}$ but not in $\mathbb{P}^{n-1} \times \mathbb{P}$ are, for instance, the codimension 2 subspace $\lambda = \mu = 0$.

In general, let $n = n_1 + \cdots + n_s$ be a partition of n. We will divide variables $x_1, \ldots, x_n$ into s sets, and write $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_s)$ for $\mathbf{x}_i \in \mathbb{C}^{n_i}$. The same convention will hold for multi-indices.

Theorem 1.5 (Multihomogeneous Bézout)**.** *Let $n = n_1 + \cdots + n_s$, with $n_1, \ldots, n_s \in \mathbb{N}$. Let $d_{ij} \in F_{\geq 0}$ be given for $1 \leq i \leq n$ and $1 \leq j \leq s$.*

Let B denote the coefficient of $\omega_1^{n_1} \omega_2^{n_2} \cdots \omega_s^{n_s}$ in

$$\prod_{i=1}^{n} \left(d_{i1}\omega_1 + \cdots + d_{is}\omega_s \right).$$

Then, for a generic choice of coefficients $f_{i\mathbf{a}} \in \mathbb{C}$, the system of equations

$$\begin{aligned} f_1(\mathbf{x}) &= \sum_{\substack{|\mathbf{a}_1| \leq d_{11} \\ \vdots \\ |\mathbf{a}_s| \leq d_{1s}}} f_{1\mathbf{a}} \mathbf{x}_1^{\mathbf{a}_1} \cdots \mathbf{x}_s^{\mathbf{a}_s} \\ &\vdots \\ f_n(\mathbf{x}) &= \sum_{\substack{|\mathbf{a}_1| \leq d_{n1} \\ \vdots \\ |\mathbf{a}_s| \leq d_{ns}}} f_{n\mathbf{a}} \mathbf{x}_1^{\mathbf{a}_1} \cdots \mathbf{x}_s^{\mathbf{a}_s} \end{aligned}$$

has exactly B roots $\mathbf{x}$ in $\mathbb{C}^n$. The number of isolated roots is never more than the number above.

This can also be formulated in terms of homogeneous polynomials and roots in multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. The above theorem is quite convenient when the partition of variables is given.

The reader should be aware that it is **NP**-hard to find, given a system, the best partition of variables [10]. Even computing an approximation of the minimal Bézout B is **NP**-hard.

For a formal proof of Theorem 1.5 see [9, Sec.5.5].

Exercise 1.2. Prove Theorem 1.5, assuming the same basic facts as in the proof of Bézout's Theorem.

3. Sparse polynomial systems

The following theorems are proved in [9].

Theorem 1.6 (Kushnirenko [8])**.** *Let $A \subset \mathbb{Z}^n$ be finite. Let $\mathcal{A}$ be the convex hull of A. Then, for a generic choice of coefficients $f_{i\mathbf{a}} \in \mathbb{C}$, the system of equations*

$$\begin{aligned} f_1(\mathbf{x}) &= \sum_{\mathbf{a}\in A} f_{1\mathbf{a}} \mathbf{x}^{\mathbf{a}} \\ &\vdots \\ f_n(\mathbf{x}) &= \sum_{\mathbf{a}\in A} f_{n\mathbf{a}} \mathbf{x}^{\mathbf{a}} \end{aligned}$$

has exactly $B = n!\mathrm{Vol}(\mathcal{A})$ roots $\mathbf{x}$ in $(\mathbb{C} \setminus \{0\})^n$. The number of isolated roots is never more than B.

The case $n = 1$ was known to Newton, and $n = 2$ was published by Minding [11] in 1841.

We call A the *support* of equations $f_1, \dots, f_n$. When each equation has a different support, root counting requires a more subtle statement.

Definition 1.7 (Minkowski linear combinations)**.** (See fig.1) Given convex sets $\mathcal{A}_1, \dots, \mathcal{A}_n$ and fixed coefficients $\lambda_1, \dots, \lambda_n$, the linear combination $\lambda_1\mathcal{A}_1 + \cdots + \lambda_n\mathcal{A}_n$ is the set of all

$$\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n$$

where $\mathbf{a}_i \in \mathcal{A}_i$.

The reader will show in the exercises that

Proposition 1.8. *Let $\mathcal{A}_1, \dots, \mathcal{A}_s$ be compact convex subsets of $\mathbb{R}^n$. Let $\lambda_1, \dots, \lambda_s > 0$. Then,*

$$\mathrm{Vol}(\lambda_1\mathcal{A}_1 + \cdots + \lambda_s\mathcal{A}_s)$$

is a homogeneous polynomial of degree s in $\lambda_1, \dots, \lambda_s$.

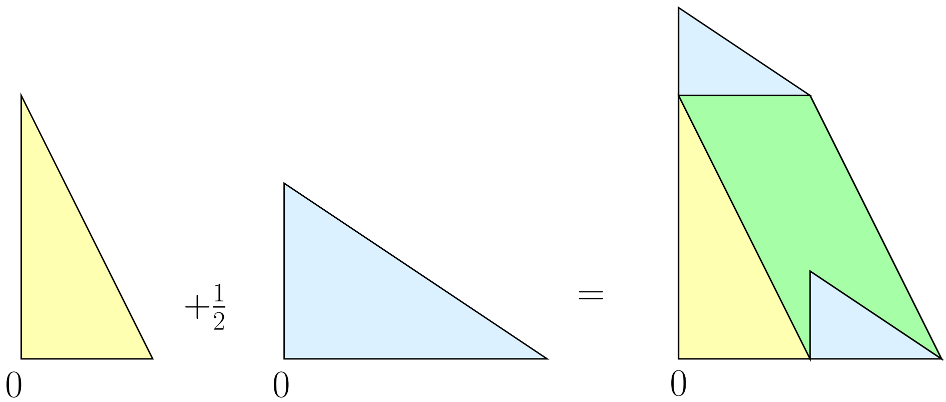

FIGURE 1. Minkowski linear combination.

Theorem 1.9 (Bernstein [2]). *Let $A_1, \dots, A_n \subset \mathbb{Z}^n$ be finite sets. Let $\mathcal{A}_i$ be the convex hull of A_i. Let B be the coefficient of $\lambda_1 \dots \lambda_n$ in the polynomial*

$$\mathrm{Vol}(\lambda_1 \mathcal{A}_1 + \cdots + \lambda_n \mathcal{A}_n).$$

Then, for a generic choice of coefficients $f_{i\mathbf{a}} \in \mathbb{C}$, the system of equations

$$\begin{aligned} f_1(\mathbf{x}) &= \sum_{\mathbf{a} \in A_1} f_{1\mathbf{a}} \mathbf{x}^{\mathbf{a}} \\ &\vdots \\ f_n(\mathbf{x}) &= \sum_{\mathbf{a} \in A_n} f_{n\mathbf{a}} \mathbf{x}^{\mathbf{a}} \end{aligned}$$

has exactly B roots $\mathbf{x}$ in $(\mathbb{C} \setminus \{0\})^n$. The number of isolated roots is never more than B.

The number $B/n!$ is known as the *mixed volume* of $\mathcal{A}_1, \dots, \mathcal{A}_n$. The generic root number B is also known as the BKK bound, after Bernstein, Kushnirenko and Khovanskii [3].

The objective of the Exercises below is to show Proposition 1.8. We will show it first for $s = 2$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be compact convex subsets of $\mathbb{R}^n$. Let E_i denote the linear hull of A_i, and assume without loss of generality that 0 is in the interior of A_i as a subset of E_i.

For any point $x \in \mathcal{A}_1$, define the cone x^C as the set of all $y \in E_2$ with the following property: for all $x' \in A_1$, $\langle y, x - x' \rangle \geq 0$.

Exercise 1.3. Let $\lambda_1, \lambda_2 > 0$ and $\mathcal{A} = \lambda_1 \mathcal{A}_1 + \lambda_2 \mathcal{A}_2$. Show that for all $z \in A$, there are $x \in \mathcal{A}_1$, $y \in x^C \cap \mathcal{A}_2$ such that $z = \lambda_1 x + \lambda_2 y$.

Exercise 1.4. Show that this decomposition is unique.

Exercise 1.5. Assume that λ_1 and λ_2 are fixed. Show that the map $z \mapsto (x, y)$ given by the decomposition above is Lipschitz.

At this point you need to believe the following fact.

Theorem 1.10 (Rademacher). *Let U be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}^m$ be Lipschitz. Then f is smooth, except possibly on a measure zero subset.*

Exercise 1.6. Use Rademacher's theorem to show that $z \mapsto (x, y)$ is smooth almost everywhere. Can you give a description of the non-smoothness set?

Exercise 1.7. Conclude the proof of Proposition 1.8 with $s = 2$.

Exercise 1.8. Generalize for all values of s.

4. Smale's 17th problem

Theorems like Bézout's or Bernstein's give precise information on the solution of systems of polynomial equations. Proofs of those theorems (such as in [9]) give a hint on how to find those roots. They do not necessarily help us to find those roots in an efficient way.

In this aspect, nonlinear equation solving is radically different from the subject of linear equation solving, where algorithms have running time typically bounded by a small degree polynomial on the input size. Here the number of roots is already exponential, and even finding one root can be a desperate task.

As in numerical linear algebra, nonlinear systems of equations may have solutions that are extremely sensitive to the value of the coefficients. Instances with such behavior are said to be *poorly conditioned*, and their 'hardness' is measured by an invariant known as the *condition number*. It is known that the condition number of random polynomial systems is small with high probability (See [9]Ch.8). Smale 17^{th} problem was introduced in [12] as:

Open Problem 1.11 (Smale). *Can a zero of n complex polynomial equations in n unknowns be found approximately , on the average, in polynomial time with a uniform algorithm?*

The precise probability space referred in [12] is what we call $(\mathcal{H}_{\mathbf{d}}, \mathrm{d}\mathcal{H}_{\mathbf{d}})$. Zero means a zero in projective space $\mathbb{P}^n$, and the notion of approximate zero is discussed in [4, 9]. Polynomial time means that the running time of the algorithm should be bound by a polynomial in the input size, that we can take as $N = \dim \mathcal{H}_{\mathbf{d}}$. The precise model of computation will not be discussed in this book, and we refer to [4]. However, the algorithm should be **uniform** in the sense that the same algorithm should work for all inputs. The number n of variables and degrees $\mathbf{d} = (d_1, \dots, d_n)$ are part of the input.

Exercise 1.9. Show that $N = \sum_{i=1}^{n} \begin{pmatrix} d_i + n \\ n \end{pmatrix}$. Conclude that there cannot exist an algorithm that approximates all the roots of a random homogeneous polynomial system in polynomial time.

Lecture 2. Differential forms

Through this lecture[2], vectors are represented boldface such as $\mathbf{x}$ and coordinates are represented as x_j. Whenever we are speaking about a collection of vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$, x_{ij} is the j-th coordinate of the i-th vector.

5. Multilinear algebra over $\mathbb{R}$

Let $\mathcal{A}^k$ be the space of alternating k-forms in $\mathbb{R}^n$, that is the space of all k-linear forms $\alpha : (\mathbb{R}^n)^k \to \mathbb{R}$ such that, for all permutation

[2]This is an edited version of Chapter 4 in Gregorio Malajovich, *Nonlinear Equations*, Publicações Matemáticas do IMPA – 28º Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro

$\sigma \in S_k$ (the permutation group of k elements), we have:

$$\alpha(\mathbf{u}_{\sigma_1}, \ldots, \mathbf{u}_{\sigma_k}) = (-1)^{|\sigma|}\alpha(\mathbf{u}_1, \ldots, \mathbf{u}_k).$$

Above, $|\sigma|$ is minimal so that σ is the composition of $|\sigma|$ *elementary permutations* (permutations fixing all elements but two).

The *canonical basis* of $\mathcal{A}^k$ is given by the forms $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, defined by

$$\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \sum_{\sigma \in S_k} (-1)^{|\sigma|} u_{\sigma(1)i_1} u_{\sigma(2)i_2} \cdots u_{\sigma(k)i_k}.$$

The *wedge product* $\wedge : \mathcal{A}^k \times \mathcal{A}^l \to \mathcal{A}^{k+l}$ is defined by

$$\alpha \wedge \beta \ (\mathbf{u}_1, \ldots, \mathbf{u}_{k+l}) =$$
$$= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{|\sigma|} \alpha(\mathbf{u}_{\sigma(1)}, \ldots, \mathbf{u}_{\sigma(k)}) \beta(\mathbf{u}_{\sigma(k+1)}, \ldots, \mathbf{u}_{\sigma(k+l)})$$

The coefficient $\frac{1}{k!l!}$ above may be replaced by $\binom{k+l}{k}$ if one replaces the sum by the anti-symmetric average over S_{k+l}. This convention makes the wedge product associative, in the sense that

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \tag{1}$$

so we just write $\alpha \wedge \beta \wedge \gamma$. This is also compatible with the notation $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$.

Another important property of the wedge product is the following: if $\alpha \in \mathcal{A}^k$ and $\beta \in \mathcal{A}^l$, then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha. \tag{2}$$

Let $U \subseteq \mathbb{R}^n$ be an open set (in the usual topology), and let $\mathcal{C}^\infty(U)$ denote the space of all smooth real valued functions defined on U. The fact that a linear k-form takes values in $\mathbb{R}$ is immaterial in all the definitions above.

Definition 2.1. The space of *differential k-forms in U*, denoted by $\mathcal{A}^k(U)$, is the space of linear k-forms defined in $\mathbb{R}^n$ with values in $\mathcal{C}^\infty(U)$.

This is equivalent to smoothly assigning to each point $\mathbf{x}$ on U, a linear k-form with values in $\mathbb{R}$. If $\alpha \in \mathcal{A}^k$, we can therefore write

$$\alpha_{\mathbf{x}} = \sum_{1 \le i_1 < \cdots < i_k \le n} \alpha_{i_1,\dots,i_k}(\mathbf{x})\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

Properties (1) and (2) hold in this context. We introduce the *exterior derivative* operator $\mathrm{d} : \mathcal{A}^k \to \mathcal{A}^{k+1}$:

$$\mathrm{d}\alpha_{\mathbf{x}} = \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j \le n \\ j \ne i_1,\dots,i_k}} \frac{\partial \alpha_{i_1,\dots,i_k}}{\partial x_j}(\mathbf{x})\, \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

Setting $\mathcal{A}^0(U) = \mathcal{C}^\infty(U)$, we see that d coincides with the ordinary derivative of functions. The exterior derivative is $\mathbb{R}$-linear and furthermore

$$\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0 \tag{3}$$

and

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta \tag{4}$$

Definition 2.2. Let $f : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^n$ be of class $\mathcal{C}^\infty$. The *pull-back* of a differential form $\alpha \in \mathcal{A}^k(V)$ by f, denoted by $f^*\alpha$, is the element of $\mathcal{A}^k(U)$ given by

$$(f^*\alpha)_{\mathbf{x}}(\mathbf{u}_1, \dots, \mathbf{u}_k) = \alpha_{f(\mathbf{x})}\left(Df(\mathbf{x})\mathbf{u}_1, \dots, Df(\mathbf{x})\mathbf{u}_k\right).$$

The chain rule for functions can be written simply as

$$\mathrm{d}(f \circ g) = g^*\mathrm{d}f$$

Exercise 2.1. Check formulas (1), (2), (3), (4).

Exercise 2.2. Show that if A is an $n \times n$ matrix,

$$\begin{aligned} &\det(A)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \\ &\quad = (A_{11}\mathrm{d}x_1 + \cdots + A_{1n}\mathrm{d}x_n) \wedge \cdots \wedge (A_{n1}\mathrm{d}x_1 + \cdots + A_{nn}\mathrm{d}x_n) \end{aligned}$$

6. Complex differential forms

An old tradition dictates that x means the 'thing', the unknown on one equation. While I try to comply in most of this text, here I will switch to another convention: if z is a complex number, x is its real part and y its imaginary part. This convention extends to vectors so

$$\mathbf{z} = \mathbf{x} + \sqrt{-1}\ \mathbf{y}.$$

The sets $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ may be identified by

$$\mathbf{z} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ \vdots \\ y_n \end{bmatrix}.$$

It is possible to define alternating k-forms in $\mathbb{C}^n$ as complex-valued alternating k-forms in $\mathbb{R}^{2n}$. However, this approach misses some of the structure related to the linearity over $\mathbb{C}$ and holomorphic functions. Instead, it is usual to define $\mathcal{A}^{k0}$ as the space of complex valued alternating k-forms in $\mathbb{C}^n$. A basis for $\mathcal{A}^{k0}$ is given by the expressions

$$\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_k}, \quad 1 \le i_1 < i_2 < \cdots < i_k \le n.$$

They are interpreted as

$$\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_k}(\mathbf{u}_1, \dots, \mathbf{u}_k) = \sum_{\sigma \in S_k} (-1)^{|\sigma|} u_{\sigma(1)i_1} u_{\sigma(2)i_2} \cdots u_{\sigma(k)i_k}.$$

Notice that $\mathrm{d}z_i = \mathrm{d}x_i + \sqrt{-1}\ \mathrm{d}y_i$. We may also define $\mathrm{d}\bar{z}_i = \mathrm{d}x_i - \sqrt{-1}\ \mathrm{d}y_i$. Next we define $\mathcal{A}^{kl}$ as the complex vector space spanned by all the expressions

$$\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_k} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_l}$$

for $1 \le i_1 < i_2 < \cdots < i_k \le n, 1 \le j_1 < j_2 < \cdots < j_l \le n$. Since

$$\mathrm{d}x_i \wedge \mathrm{d}y_i = -2\sqrt{-1}\ \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i,$$

the standard volume form in $\mathbb{C}^n$ is

$$\mathrm{d}V = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n = \left(\frac{\sqrt{-1}}{2}\right)^n \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n.$$

The following fact is quite useful:

Lemma 2.3. *If A is an $n \times n$ matrix, then*

$$|\det(A)|^2 \; \mathrm{d}V = \bigwedge_{k=1}^{n} \sum_{i,j=1}^{n} \frac{\sqrt{-1}}{2} A_{ki} \bar{A}_{kj} \; \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$

Proof. As in exercise 2.2,

$$\det(A) \; \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = \bigwedge_{k=1}^{n} \sum_{i=1}^{n} A_{ki} \; \mathrm{d}z_i$$

and

$$\overline{\det(A)} \; \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n = \bigwedge_{k=1}^{n} \sum_{j=1}^{n} \bar{A}_{kj} \; \mathrm{d}\bar{z}_j.$$

The Lemma is proved by wedging the two expressions above and multiplying by $(\sqrt{-1}/2)^n$. □

If U is an open subset of $\mathbb{C}^n$, then $\mathcal{C}^\infty(U, \mathbb{C})$ is the *complex* space of all smooth complex valued functions of U. Here, *smooth* means of class $\mathcal{C}^\infty$ and *real* derivatives are assumed. The holomorphic and anti-holomorphic derivatives are defined as

$$\frac{\partial f}{\partial z_i} = \frac{1}{2} \left(\frac{\partial f}{\partial x_i} - \sqrt{-1} \frac{\partial f}{\partial y_i} \right)$$

and

$$\frac{\partial f}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial f}{\partial x_i} + \sqrt{-1} \frac{\partial f}{\partial y_i} \right)$$

The Cauchy-Riemann equations for a function f to be holomorphic are just

$$\frac{\partial f}{\partial \bar{z}_i} = 0.$$

We denote by $\partial : \mathcal{A}^{kl}(U) \to \mathcal{A}^{k+1,l}(U)$ the holomorphic differential, and by $\bar{\partial} : \mathcal{A}^{kl}(U) \to \mathcal{A}^{k,l+1}(U)$ the anti-holomorphic differential. If

$$\alpha(\mathbf{z}) = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_k \le n \\ 1 \le j_1 < j_2 < \cdots < j_l \le n}} \alpha_{i_1, \dots, j_l}(\mathbf{z}) \; \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_l},$$

then

$$\partial\alpha(\mathbf{z}) = \sum_{\substack{1\le i_1<i_2<\cdots<i_k\le n \\ 1\le j_1<j_2<\cdots<j_l\le n \\ 1\le k\le n,\ k\ne i_r}} \frac{\partial \alpha_{i_1,\dots,j_l}}{\partial z_k}(\mathbf{z})\ \mathrm{d}z_k \wedge \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_l},$$

and

$$\bar\partial\alpha(\mathbf{z}) = \sum_{\substack{1\le i_1<i_2<\cdots<i_k\le n \\ 1\le j_1<j_2<\cdots<j_l\le n \\ 1\le k\le n,\ k\ne j_r}} \frac{\partial \alpha_{i_1,\dots,j_l}}{\partial \bar{z}_k}(\mathbf{z})\ \mathrm{d}\bar{z}_k \wedge \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_l},$$

The *total* differential is $\mathrm{d} = \partial + \bar\partial$. Another useful fact is that $\partial^2 = \bar\partial^2 = 0$.

7. Kähler geometry

Let $U \subseteq \mathbb{C}^n$ be an open set, and let $g_{ij} : U \to \mathbb{C}$ be such that each $g_{ij} \in \mathcal{C}^\infty(U, \mathbb{C}^n)$ and furthermore, the matrix $g(z) = [g_{ij}(z)]_{1\le i,j\le n}$ is Hermitian positive definite at each z.

This defines, at each $\mathbf{z} \in U$, the Hermitian inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{z}} = \sum_{i,j=1}^{n} g_{ij}(\mathbf{z}) u_i \bar{v}_j .$$

The corresponding volume form is $\mathrm{d}V(\mathbf{z}) = |\det g_{ij}(\mathbf{z})|$ (compare with the Riemannian case).

Because $g(\mathbf{z})$ is Hermitian, its real part is symmetric and defines a Riemannian metric. Thus $\omega_{\mathbf{z}} = -\mathrm{im}\ g_{\mathbf{z}}(\cdot,\cdot)$ is skew-symmetric whence in $\mathcal{A}^{11}(U)$.

Definition 2.4. A *Kähler form* is a form $\omega_{\mathbf{z}} \in \mathcal{A}^{11}(U)$ that is:

(1) *positive*:
$$\omega_{\mathbf{z}}(\mathbf{u}, \sqrt{-1}\ \mathbf{u}) \ge 0$$
with equality only if $\mathbf{u} = 0$, and

(2) *closed:* $\mathrm{d}\omega_{\mathbf{z}} \equiv 0$.

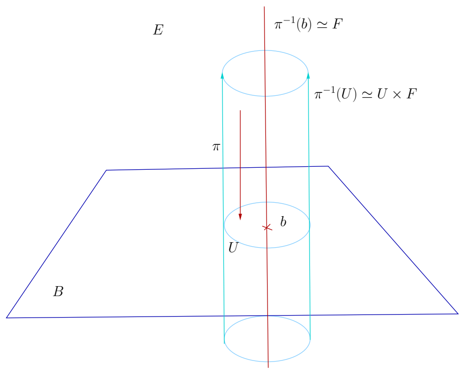

FIGURE 2. Fiber bundle.

The canonical Kähler form in $\mathbb{C}^n$ is

$$\omega = \frac{\sqrt{-1}}{2}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \frac{\sqrt{-1}}{2}\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 + \cdots + \frac{\sqrt{-1}}{2}\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n.$$

Given a Kähler form, its volume form can be written as

$$\mathrm{d}V_{\mathbf{z}} = \frac{1}{n!}\underbrace{\omega_{\mathbf{z}} \wedge \omega_{\mathbf{z}} \wedge \cdots \wedge \omega_{\mathbf{z}}}_{n \text{ times}}.$$

The definition above is for a Kähler structure on a subset of $\mathbb{C}^n$. This definition can be extended to a complex manifold, or to a $2n$-manifold where a 'complex multiplication' $J : T_{\mathbf{z}}M \to T_{\mathbf{z}}M$, $J^2 = -I$, is defined.

An amazing fact about Kähler manifolds is the following.

Theorem 2.5 (Wirtinger)**.** *Let S be a d-dimensional complex submanifold of a Kähler manifold M. Then it inherits its Kähler form, and*

$$\mathrm{Vol}(S) = \frac{1}{d!}\int_S \underbrace{\omega_{\mathbf{z}} \wedge \cdots \wedge \omega_{\mathbf{z}}}_{d \textit{ times}}.$$

Since ω is a closed form, $\omega \wedge \cdots \wedge \omega$ is also closed. When S happens to be a boundary, its volume is zero.

8. The co-area formula

Definition 2.6. A smooth (real, complex) fiber bundle is a tuple (E, B, π, F) such that

(1) E is a smooth (real, complex) manifold (known as *total space*).
(2) B is a smooth (real , complex) manifold (known as *base space*).
(3) $\pi : E \mapsto B$ is a smooth surjection (the *projection*).
(4) F is a (real, complex) smooth manifold (the *fiber*).
(5) The local triviality condition: for every $p \in E$, there is an open neighborhood $U \ni \pi(p)$ in B and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times F$. (the *local trivialization*).
(6) Moreover, $\Phi_{|\pi^{-1}\circ\pi(p)} \to F$ is a diffeomorphism.

(See figure 2).

Familiar examples of fiber bundles are the tangent bundle of a manifold, the normal bundle of an embedded manifold, etc... In those case the fiber is a vector space, so we speak of a *vector bundle.* The fiber may be endowed of another structure (say a group) which is immaterial here.

Here is a less familiar example of a vector bundle. Recall that $\mathcal{P}_d$ is the space of complex univariate polynomials of degree $\leq d$. Let $\mathcal{V} = \{(f, x) \in \mathcal{P}_d \times \mathbb{C} : f(x) = 0\}$. This set is known as the *solution variety.* Let $\pi_2 : \mathcal{V} \to \mathbb{C}$ be the projection into the second set of coordinates, namely $\pi_2(f, x) = x$. Then $\pi_2 : \mathcal{V} \to \mathbb{C}$ is a vector bundle.

The co-area formula is a Fubini-type theorem for fiber bundles:

Theorem 2.7 (co-area formula)**.** *Let (E, B, π, F) be a real smooth fiber bundle. Assume that B is finite dimensional. Let $f : E \to \mathbb{R}_{\geq 0}$ be measurable. Then whenever the left integral exists,*

$$\int_E f(p)\mathrm{d}E(p) = \int_B \mathrm{d}B(x) \int_{E_x} (\det D\pi(p) D\pi(p)^*)^{-1/2} f(p)\mathrm{d}E_x(p).$$

with $E_x = \pi^{-1}(x)$.

Lemma 2.8. *In the conditions of Theorem 2.7, there is a locally finite open covering $\mathcal{U} = \{U_\alpha\}$ of B, and a family of smooth functions $\psi_\alpha \geq 0$ with domain B vanishing in $B \setminus U_\alpha$ such that*

(1) *Each* $U_\alpha \in \mathfrak{U}$ *is such that there is a local trivialization* Φ *with domain* $\Phi^{-1}(U_\alpha)$*.*

(2)
$$\sum_\alpha \psi_\alpha(x) \equiv 1.$$

The family $\{\psi_\alpha\}$ is said to be a *partition of unity* for $\pi : E \to B$.

Proof of theorem 2.7. Let ψ_α be the partition of unity from Lemma 2.8. By replacing f by $f(\psi_\alpha \circ \pi)$ and then adding for all α, we can assume without loss of generality that f vanishes outside the domain $\pi^{-1}(U)$ of a local trivialization.

Now,

$$\begin{aligned}
\int_E f(p)\mathrm{d}E(p) &= \int_{\pi^{-1}(U)} f(p)\mathrm{d}E(p) \\
&= \int_{\Phi(\pi^{-1}(U))} \det D\Phi^{-1}(x,y) f(\Phi^{-1}(x,y))\mathrm{d}B(x)\mathrm{d}F(y) \\
&= \int_U \mathrm{d}B(x) \int_F \det D\Phi^{-1}(x,y) f(\Phi^{-1}(x,y))\mathrm{d}F(y)
\end{aligned}$$

using Fubini's theorem. Note that $\Phi_{|F_x} \to F$ is a diffeomorphism, so the inner integral can be replaced by

$$\int_{F_x} \det D\Phi_{|F_x} \det D\Phi^{-1}(p) f(p)\mathrm{d}F_x(p).$$

Moreover, by splitting $T_pE = \ker D\pi^\perp \oplus \ker D\pi$ and noticing that $F_x = \ker D\pi(p)$,

$$D\Phi = \begin{bmatrix} D\pi(p) & 0 \\ ? & D\Phi_{|F_x}(p) \end{bmatrix}.$$

Therefore

$$\det D\Phi_{|F_x} \det D\Phi^{-1} = \det \left(D\pi^{-1}_{|\ker D\pi^\perp} \right) = \left(\det D\pi D\pi^*\right)^{-1/2}.$$

□

When the fiber bundle is complex, we obtain a similar formula by assimilating $\mathbb{C}^n$ to $\mathbb{R}^{2n}$:

Theorem 2.9 (co-area formula)**.** *Let (E, B, π, F) be a complex smooth fiber bundle. Assume that B is finite dimensional. Let $f : E \to \mathbb{R}_{\geq 0}$ be measurable. Then whenever the left integral exists,*

$$\int_E f(p) \mathrm{d}E(p) = \int_B \mathrm{d}B(x) \int_{E_x} (\det D\pi(p) D\pi(p)^*)^{-1} f(p) \mathrm{d}E_x(p).$$

with $E_x = \pi^{-1}(x)$.

9. Projective space

Complex projective space $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the multiplicative group $\mathbb{C}_\times$. This means that the elements of $\mathbb{P}^n$ are complex 'lines' of the form

$$(x_0 : \cdots : x_n) = \{(\lambda x_0, \lambda x_1, \cdots, \lambda x_n) : 0 \neq \lambda \in \mathbb{C}\}.$$

It is possible to define local charts at $(p_0 : \cdots : p_n) : \mathbf{p}^\perp \subset \mathbb{C}^{n+1} \to \mathbb{P}^n$ by sending $\mathbf{x}$ into $(p_0 + x_0 : \cdots : p_n + x_n)$.

There is a canonical way to define a metric in $\mathbb{P}^n$, in such a way that for $\|\mathbf{p}\| = 1$, the chart $\mathbf{x} \mapsto \mathbf{p} + \mathbf{x}$ is a local isometry at $\mathbf{x} = 0$. Define the *Fubini-Study* differential form by

$$\omega_{\mathbf{z}} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \|\mathbf{z}\|^2. \tag{5}$$

Expanding the expression above, we get

$$\omega_{\mathbf{z}} = \frac{\sqrt{-1}}{2} \left(\frac{1}{\|\mathbf{z}\|^2} \sum_{j=0}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j - \frac{1}{\|\mathbf{z}\|^4} \sum_{j,k=0}^{n} \bar{z}_j z_k \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \right).$$

When (for instance) $\mathbf{z} = \mathrm{e}_0$,

$$\omega_{\mathrm{e}_0} = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

Similarly, if E is any complex vector space, $\mathbb{P}(E)$ is the quotient of E by $\mathbb{C}_\times$. When E admits a norm, the Fubini-Study metric in $\mathbb{P}(E)$ can be introduced in a similar way.

Proposition 2.10.

$$\mathrm{Vol}(\mathbb{P}^n) = \frac{\pi^n}{n!}.$$

Before proving Proposition 2.10, we state and prove the formula for the volume of the sphere. The *Gamma function* is defined by

$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} \, \mathrm{d}t.$$

Direct integration gives that $\Gamma(1) = 1$, and integration by parts shows that $\Gamma(r) = (r-1)\Gamma(r-1)$ so that if $n \in \mathbb{N}$, $\Gamma(n) = n - 1!$

Proposition 2.11.

$$\mathrm{Vol}(\mathbb{S}^k) = 2\frac{\pi^{(k+1)/2}}{\Gamma\left(\frac{k+1}{2}\right)}.$$

Proof. By using polar coordinates in $\mathbb{R}^{k+1}$, we can infer the following expression for the integral of the Gaussian normal:

$$\begin{aligned}
\int_{\mathbb{R}^{k+1}} \frac{1}{\sqrt{2\pi}^{k+1}} e^{-\|\mathbf{x}\|^2/2} \, \mathrm{d}V_{\mathbf{x}} &= \int_{S^k} \mathrm{d}S^k(\Theta) \int_0^\infty \frac{R^k}{\sqrt{2\pi}^{k+1}} e^{-R^2/2} \, \mathrm{d}R \\
&= \mathrm{Vol}(S^k) \int_0^\infty \frac{r^{(k-1)/2}}{2\sqrt{\pi}^{k+1}} e^{-r} \, \mathrm{d}r \\
&= \mathrm{Vol}(S^k) \frac{\Gamma\left(\frac{k+1}{2}\right)}{2\sqrt{\pi}^{k+1}}
\end{aligned}$$

The integral on the left is just

$$\left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-x} \, \mathrm{d}x \right)^{k+1}$$

and from the case $k = 1$, we can infer that it is equal to 1. The proposition then follows for all k. □

Proof of Proposition 2.10. Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere $|\mathbf{z}| = 1$. The *Hopf fibration* is the natural projection of S^{2n+1} onto $\mathbb{P}^n$. The preimage of any $(z_0 : \cdots : z_n)$ is always a great circle in S^{2n+1}.

We claim that

$$\mathrm{Vol}(\mathbb{P}^n) = \frac{1}{2\pi}\mathrm{Vol}(S^{2n+1}).$$

Since we know that the right-hand-term is $\pi^n/n!$, this will prove the Proposition.

The unitary group $U(n+1)$ acts on $\mathbb{C}^{n+1}_{\neq 0}$ by $Q, \mathbf{x} \mapsto Q\mathbf{x}$. This induces transitive actions in $\mathbb{P}^n$ and S^{2n+1}. Moreover, if $\|\mathbf{x}\| = 1$,

$$H(Q\mathbf{x}) = Q(x_0 : \cdots : x_n)$$

so $DH_{Q\mathbf{x}} = QDH_{\mathbf{x}}$. It follows that the Normal Jacobian $\det(DHDH^*)$ is invariant by $U(n+1)$-action, and we may compute it at a single point, say at e_0. Recall our convention $z_i = x_i + \sqrt{-1}\, y_i$. The tangent space $T_{\mathrm{e}_0}S^n$ has coordinates $y_0, x_1, y_1, \ldots, y_n$ while the tangent space $T_{(1:0:\cdots:0)}\mathbb{P}^n$ has coordinates $x_1, y_1, \ldots, y_n$. With those coordinates,

$$DH(\mathrm{e}_0) = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

(white spaces are zeros). Thus $DH(\mathrm{e}_0)\ DH(\mathrm{e}_0)^*$ is the identity.

The co-area formula (Theorem 2.7) now reads:

$$\begin{aligned} \mathrm{Vol} S^{2n+1} &= \int_{S^{2n+1}} \mathrm{d}S^{2n+1} \\ &= \int_{\mathbb{P}^n} d\mathbb{P}^n(\mathbf{x}) \int_{H^{-1}(\mathbf{x})} |\det(DH(\mathbf{y})\ DH^*(\mathbf{y}))|^{-1} \mathrm{d}S^1(\mathbf{y}) \\ &= 2\pi \mathrm{Vol}(\mathbb{P}^n) \end{aligned}$$

□

We come now to another consequence of Wirtinger's theorem. Let W be a variety (irreducible Zariski closed set) of complex dimension k in $\mathbb{P}^n$. It follows from Noether's normalization Lemma that the intersection of W with a generic plane Π of dimension $n-k$ is precisely d points.

We change coordinates so that Π is the plane $y_{k+1} = \cdots = y_n = 0$. Let $P = \{(y_0 : \cdots : y_k : 0 : \cdots 0)\}$ be a copy of $\mathbb{P}^k$. Then consider the formal sum (k-chain) $W - dP$. This is precisely the boundary of the $k+1$-chain

$$\mathcal{D} = \{(y_0 : \cdots : y_k : ty_{k+1} : \cdots : ty_n) : \mathbf{y} \in W, t \in [0,1]\}.$$

By Wirtinger's theorem (Th. 2.5), $W - dP$ has zero volume. We conclude that

Theorem 2.12. *Let $W \subset \mathbb{P}^n$ be a variety of dimension k and degree d. Then,*

$$\mathrm{Vol}\ W = d\frac{\pi^k}{k!}.$$

Remark 2.13. Many authors such as [6] divide the Fubini-Study metric by π. This is a neat convention, because it makes the volume of $\mathbb{P}^n$ equal to $1/n!$. However, this conflicts with the notations used in the subject of polynomial equation solving (such as in [4]), so I opt here for maintaining the notational integrity of the subject.

Lecture 3. Reproducing kernel spaces

In this lecture [3] we introduce a general formalism that allows to generalize the classical theorems of Bézout, Kushnirenko and Bernstein. Also, we deduce the Bézout theorem from the root density integral.

10. Fewspaces

Let M be an n-dimensional complex manifold. Our main object of study in this book are the systems of equations

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0,$$

where $f_i \in \mathcal{F}_i$, and $\mathcal{F}_i$ is a suitable Hilbert space whose elements are functions from M to $\mathbb{C}$.

Main examples for M are $\mathbb{C}^n$, $(\mathbb{C}_{\neq 0})^n$, a 'quotient manifold' such as $\mathbb{C}^n/(2\pi\sqrt{-1}\ \mathbb{Z}^n)$, a polydisk $|z_1|, \ldots, |z_n| < 1$, or a n-dimensional

[3]This is an edited version of Chapter 4 in Gregorio Malajovich, *Nonlinear Equations*, Publicações Matemáticas do IMPA – 28º Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro

quasi-affine variety in $\mathbb{C}^n$. Examples of $\mathcal{F}_i$ are the space of polynomials of degree $\leq d_i$ for a certain d_i, or spaces spanned by a finite collection of arbitrary holomorphic functions.

It may be convenient to consider the f_i's as either given or random. By *random* we mean that the f_i are independently normally distributed random variables with unit variance.

Remark 3.1. The definition and main properties of holomorphic functions on several variables follow, in general lines, the main ideas from one complex variable. The unaware reader may want to read chapter 0 and maybe chapter 1 in [7] before proceeding. Regarding reproducing kernel spaces, a canonical reference is Aronszajn's paper [1]

The aim of this chapter is to define what sort of spaces are 'acceptable' for the problem above. Most of functional analysis deals with spaces that are made large enough to contain certain objects. In contrast, we need to avoid 'large' spaces if we want to *count* roots.

The general theory will include equations on quotient manifolds, such as homogeneous polynomials on projective space. We start with the simpler definition, where the equations are actual functions. (See [9] for a more general theory).

Definition 3.2. A *fewnomial space* (or *fewspace* for short) of functions over a complex manifold M is a Hilbert space of holomorphic functions from M to $\mathbb{C}$ such that the following holds. Let $V : M \to \mathcal{F}^*$ denote the *evaluation form* $V(\mathbf{x}) : f \mapsto f(\mathbf{x})$. For any $\mathbf{x} \in M$,

(1) $V(\mathbf{x})$ is continuous as a linear form.
(2) $V(\mathbf{x})$ is not the zero form.

In addition, we say that the fewspace is *non-degenerate* if and only if, for any $\mathbf{x} \in M$,

3. $P_{V(\mathbf{x})} DV(\mathbf{x})$ has full rank,

where P_W denotes the orthogonal projection onto $W^\perp$. (The derivative is with respect to $\mathbf{x}$). In particular, a non-degenerate fewspace has dimension $\geq n + 1$.

We say that a fewspace $\mathcal{F}$ is $\mathcal{L}^2$ if its elements have finite $\mathcal{L}^2$ norm. In this case the $\mathcal{L}^2$ inner product is assumed.

Example 3.3. Let M be an open connected subset of $\mathbb{C}^n$. *Bergman space* $\mathcal{A}(M)$ is the space of holomorphic functions defined in M with finite $\mathcal{L}^2$ norm. When M is bounded, it contains constant and linear functions, hence M is clearly a non-degenerate fewspace.

Remark 3.4. Condition 1 holds trivially for any finite dimensional fewnomial space, and less trivially for subspaces of Bergman space. (Exercise 3.1). Condition 2 may be obtained by removing points from M.

To each fewspace $\mathcal{F}$ we associate two objects: The *reproducing kernel* $K(\mathbf{x}, \mathbf{y})$ and a possibly degenerate Kähler form ω on M.

Item (1) in the definition makes $V(x)$ an element of the dual space $\mathcal{F}^*$ of $\mathcal{F}$ (more precisely, the 'continuous' dual space or space of continuous functionals). Here is a classical result about Hilbert spaces:

Theorem 3.5 (Riesz-Fréchet)**.** *Let* $\mathbb{H}$ *be a Hilbert space. If* $\phi \in \mathbb{H}^*$*, then there is a unique* $\mathbf{f} \in \mathbb{H}$ *such that*

$$\phi(\mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbb{H}} \qquad \forall \mathbf{v} \in \mathbb{H}.$$

Moreover, $\|\mathbf{f}\|_{\mathbb{H}} = \|\phi\|_{\mathbb{H}^*}$

For a proof, see [5] Th.V.5 p.81. Riesz-Fréchet representation Theorem allows to identify $\mathcal{F}$ and $\mathcal{F}^*$, whence the Kernel $K(\mathbf{x}, \mathbf{y}) = \overline{(V(\mathbf{x})^*)(\mathbf{y})}$. As a function of $\bar{\mathbf{y}}$, $K(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ for all $\mathbf{x}$.

By construction, for $f \in \mathcal{F}$,

$$f(\mathbf{y}) = \langle f(\cdot), K(\cdot, \mathbf{y}) \rangle.$$

There are two consequences. First of all,

$$K(\mathbf{y}, \mathbf{x}) = \langle K(\cdot, \mathbf{x}), K(\cdot, \mathbf{y}) \rangle = \overline{\langle K(\cdot, \mathbf{y}), K(\cdot, \mathbf{x}) \rangle} = \overline{K(\mathbf{x}, \mathbf{y})}$$

and in particular, for any fixed $\mathbf{y}$, $\mathbf{x} \mapsto K(\mathbf{x}, \mathbf{y})$ is an element of $\mathcal{F}$. Thus, $K(\mathbf{x}, \mathbf{y})$ is analytic in $\mathbf{x}$ and in $\bar{\mathbf{y}}$. Moreover, $\|K(\mathbf{x}, \cdot)\|^2 = K(\mathbf{x}, \mathbf{x})$.

Secondly, $Df(\mathbf{y})\dot{\mathbf{y}} = \langle f(\cdot), D_{\bar{\mathbf{y}}} K(\cdot, \mathbf{y}) \bar{\dot{\mathbf{y}}} \rangle$ and the same holds for higher derivatives.

Exercise 3.1. Show that V is continuous in Bergman space $\mathcal{A}(M)$. Hint: verify first that for u harmonic and r small enough,

$$\frac{1}{\text{Vol } B(\mathbf{p}, r)} \int_{B(\mathbf{p},r)} u(\mathbf{z}) \, \mathrm{d}\mathbf{z} = u(\mathbf{p}).$$

11. Metric structure on root space

Because of Definition 3.2(2), $K(\cdot, y) \neq 0$. Thus, $y \mapsto K(\cdot, y)$ induces a map from M to $\mathbb{P}(\mathcal{F})$. The differential form ω is defined as the pull-back of the Fubini-Study form ω_f of $\mathbb{P}(\mathcal{F})$ by $y \mapsto K(\cdot, y)$.

Recall from (5) that The Fubini-Study differential 1-1 form in $\mathcal{F} \setminus \{0\}$ is defined by

$$\omega_f = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \|f\|^2$$

and is equivariant by scaling. Its pull-back is

$$\omega_x = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log K(x, x).$$

When the form ω is non-degenerate for all $x \in M$, it induces a Hermitian structure on M. This happens if and only if the fewspace is a non-degenerate fewspace.

Remark 3.6. If $\mathcal{F}$ is the Bergman space, the kernel obtained above is known as the *Bergman Kernel* and the metric induced by ω as the *Bergman metric*.

Remark 3.7. If $\phi_i(x)$ denotes an orthonormal basis of $\mathcal{F}$ (finite or infinite), then the kernel can be written as

$$K(\mathbf{x}, \mathbf{y}) = \sum \phi_i(\mathbf{x}) \overline{\phi_i(\mathbf{y})}.$$

Remark 3.8. The form ω induces an element of the cohomology ring $H^*(M)$, namely the operator that takes a $2k$-chain C to $\int_C \omega \wedge \cdots \wedge \omega$.

If $\mathcal{F}$ is a fewspace and $\mathbf{x} \in M$, we denote by $\mathcal{F}_x$ the space $K(\cdot, \mathbf{x})^\perp$ of all $f \in \mathcal{F}$ vanishing at $\mathbf{x}$.

Proposition 3.9. *Let $\mathcal{F}$ be a fewspace. Let $\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbf{x}} = \omega_{\mathbf{x}}(\mathbf{u}, J\mathbf{w})$ be the (possibly degenerate) Hermitian product associated to ω. Then,*

$$\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbf{x}} = \frac{1}{2} \int_{\mathcal{F}_{\mathbf{x}}} \frac{(Df(\mathbf{x})\mathbf{u})\overline{Df(\mathbf{x})\mathbf{w}}}{K(\mathbf{x}, \mathbf{x})} \, \mathrm{d}\mathcal{F}_{\mathbf{x}} \tag{6}$$

where $\mathrm{d}\mathcal{F}_{\mathbf{x}} = \frac{1}{(2\pi)^{\dim \mathcal{F}_{\mathbf{x}}}} e^{-\|f\|^2} \mathrm{d}\lambda(f)$ *is the zero-average, unit variance Gaussian probability distribution on* $\mathcal{F}_{\mathbf{x}}$.

Proof. Let

$$P_{\mathbf{x}} = I - \frac{K(\cdot, \mathbf{x})K(\cdot, \mathbf{x})^*}{K(\mathbf{x}, \mathbf{x})}$$

be the orthogonal projection $\mathcal{F} \to \mathcal{F}_{\mathbf{x}}$. We can write the left-hand-side as:

$$\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbf{x}} \quad = \quad \frac{\langle P_{\mathbf{x}} DK(\cdot, \mathbf{x})\mathbf{u}, P_{\mathbf{x}} DK(\cdot, \mathbf{x})\mathbf{w} \rangle}{K(\mathbf{x}, \mathbf{x})}$$

For the right-hand-side, note that

$$Df(\mathbf{x})\mathbf{u} = \langle f(\cdot), DK(\cdot, \mathbf{x})\mathbf{u} \rangle = \langle f(\cdot), P_{\mathbf{x}} DK(\cdot, \mathbf{x})\mathbf{u} \rangle.$$

Let $\mathbf{U} = \frac{1}{\|K(\cdot, \mathbf{x})\|} P_{\mathbf{x}} DK(\cdot, \mathbf{x})\mathbf{u}$ and $\mathbf{W} = \frac{1}{\|K(\cdot, \mathbf{x})\|} P_{\mathbf{x}} DK(\cdot, \mathbf{x})\mathbf{w}$. Both $\mathbf{U}$ and $\mathbf{W}$ belong to $\mathcal{F}_{\mathbf{x}}$. The right-hand-side is

$$\begin{aligned}
\frac{1}{2} \int_{\mathcal{F}_{\mathbf{x}}} \frac{(Df(\mathbf{x})\mathbf{u})\overline{Df(\mathbf{x})\mathbf{w}}}{\|K(\mathbf{x}, \mathbf{x})\|^2} \, \mathrm{d}\mathcal{F}_{\mathbf{x}} \quad &= \quad \frac{1}{2} \int_{\mathcal{F}_{\mathbf{x}}} \langle \mathbf{f}, \mathbf{U} \rangle \overline{\langle f, \mathbf{W} \rangle} \, \mathrm{d}\mathcal{F}_{\mathbf{x}} \\
&= \quad \frac{1}{2} \langle \mathbf{U}, \mathbf{W} \rangle \int_{\mathbb{C}} \frac{1}{2\pi} |z|^2 e^{-|z|^2/2} \, \mathrm{d}z \\
&= \quad \langle \mathbf{U}, \mathbf{W} \rangle
\end{aligned}$$

which is equal to the left-hand-side. □

For further reference, we state that:

Lemma 3.10. *The metric coefficients g_{ij} associated to the (possibly degenerate) inner product above are*

$$g_{ij}(\mathbf{x}) = \frac{1}{K(\mathbf{x}, \mathbf{x})} \left(K_{ij}(\mathbf{x}, \mathbf{x}) - \frac{K_{i\cdot}(\mathbf{x}, \mathbf{x})K_{\cdot j}(\mathbf{x}, \mathbf{x})}{K(\mathbf{x}, \mathbf{x})} \right)$$

with the notation $K_{i\cdot}(\mathbf{x},\mathbf{y}) = \frac{\partial}{\partial x_i} K(\mathbf{x},\mathbf{y})$, $K_{\cdot j}(\mathbf{x},\mathbf{y}) = \frac{\partial}{\partial \bar{y}_j} K(\mathbf{x},\mathbf{y})$ *and* $K_{ij}(\mathbf{x},\mathbf{y}) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial \bar{y}_j} K(\mathbf{x},\mathbf{y})$.

The Fubini 1-1 form is then:

$$\omega = \frac{\sqrt{-1}}{2} \sum_{ij} g_{ij} dz_i \wedge d\bar{z}_j$$

and the volume element is $\frac{1}{n!} \bigwedge_{i=1}^{n} \omega$.

Exercise 3.2. Prove Lemma 3.10.

12. Root density

We can deduce the famous theorems by Bézout, Kushnirenko and Bernstein from the statement below. Recall that $n_{\mathcal{K}}(f)$ is the number of isolated zeros of f that belong to $\mathcal{K}$.

Theorem 3.11 (Root density). *Let* $\mathcal{K}$ *be a locally measurable set of an* n*-dimensional manifold* M*. Let* $\mathcal{F}_1, \dots, \mathcal{F}_n$ *be fewspaces. Let* $\omega_1, \dots, \omega_n$ *be the induced symplectic forms on* M*. Assume that* $\mathbf{f} = f_1, \dots, f_n$ *is a zero average, unit variance variable in* $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$*. Then,*

$$\mathbb{E}(n_{\mathcal{K}}(\mathbf{f})) = \frac{1}{\pi^n} \int_{\mathcal{K}} \omega_1 \wedge \cdots \wedge \omega_n.$$

Proof of Theorem 3.11. Let $\mathcal{V} \subset \mathcal{F} \times M$, where $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$ be the *incidence locus*, $\mathcal{V} \overset{\text{def}}{=} \{(\mathbf{f}, x) : \mathbf{f}(x) = 0\}$. (It is a variety when M is a variety). Let $\pi_1 : \mathcal{V} \to \mathcal{F}$ and $\pi_2 : \mathcal{V} \to M$ be the canonical projections.

For each $\mathbf{x} \in M$, denote by $\mathcal{F}_{\mathbf{x}} = \{\mathbf{f} \in \mathcal{F} : \mathbf{f}(\mathbf{x}) = 0\}$. Then $\mathcal{F}_{\mathbf{x}}$ is a linear space of codimension n in $\mathcal{F}$. More explicitly,

$$F_{\mathbf{x}} = K_1(\cdot, \mathbf{x})^{\perp} \times \cdots \times K_n(\cdot, \mathbf{x})^{\perp} \subset \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$$

using the notation K_i for the reproducing kernel associated to $\mathcal{F}_i$.

Let $O \in M$ be an arbitrary particular point, and let $F = \mathcal{F}_O$.

We claim that $(\mathcal{V}, M, \pi_2, F)$ is a vector bundle.

First, we should check that $\mathcal{V}$ is a manifold. Indeed, $\mathcal{V}$ is defined implicitly as $\mathrm{ev}^{-1}(0)$, where $\mathrm{ev}(\mathbf{f}, \mathbf{x}) = \mathbf{f}(\mathbf{x})$ is the evaluation function.

Let $p = (\mathbf{f}, \mathbf{x}) \in \mathcal{V}$ be given. The differential of the evaluation function at p is

$$D\mathrm{ev}(p) : \dot{\mathbf{f}}, \dot{\mathbf{x}} \mapsto D\mathbf{f}(\mathbf{x})\dot{\mathbf{x}} + \dot{\mathbf{f}}(\mathbf{x}).$$

Let us prove that $D\mathrm{ev}(p)$ has rank n.

$$D\mathrm{ev}(p)(\dot{\mathbf{f}}, 0) = \begin{bmatrix} \langle \dot{f}_1(\cdot), K_1(\cdot, \mathbf{x}) \rangle_{\mathcal{F}_1} \\ \vdots \\ \langle \dot{f}_n(\cdot), K_n(\cdot, \mathbf{x}) \rangle_{\mathcal{F}_n} \end{bmatrix}$$

and in particular, $D\mathrm{ev}(p)(\mathrm{e}_i K_i(\mathbf{x}, \cdot)/K_i(\mathbf{x}, \mathbf{x}), 0) = \mathrm{e}_i$. Therefore 0 is a regular value of ev and hence $\mathcal{V}$ is an embedded manifold.

Now, we should produce a local trivialization. Let U be a neighborhood of $\mathbf{x}$. Let $i_O : \mathcal{F}_{\mathbf{x}} \to F$ be a linear isomorphism. For $\mathbf{y} \in U$, we define $i_{\mathbf{y}} : \mathcal{F}_{\mathbf{y}} \to \mathcal{F}_{\mathbf{x}}$ by othogonal projection in each component. The neighborhood U should be chosen so that $i_{\mathbf{y}}$ is always a linear isomorphism. Explicitly,

$$\begin{aligned} i_{\mathbf{y}} = I_{\mathcal{F}_1} - &\frac{1}{K_1(\mathbf{x}, \mathbf{x})} K_1(\mathbf{x}, \cdot) K_1(\mathbf{x}, \cdot)^* \oplus \cdots \\ &\oplus I_{\mathcal{F}_n} - \frac{1}{K_n(\mathbf{x}, \mathbf{x})} K_n(\mathbf{x}, \cdot) K_n(\mathbf{x}, \cdot)^* \end{aligned}$$

so $U = \{\mathbf{y} : K_j(\mathbf{y}, \mathbf{x}) \neq 0 \ \forall j\}$.

For $q = (\mathbf{g}, \mathbf{y}) \in \pi_2^{-1}(\mathbf{x})$, set

$$\Phi(q) = (\pi_2(q), i_O \circ i_{\mathbf{y}} \circ \pi_1(q)).$$

This is clearly a diffeomorphism.

The expected number of roots of $\mathcal{F}$ is

$$\mathbb{E}(n_{\mathcal{K}}(f)) = \int_{\mathcal{V}} \chi_{\pi_2^{-1}(\mathcal{K})}(p)(\pi_1^* \mathrm{d}\mathcal{F})(p).$$

Denote by $\mathrm{d}\mathcal{F}$, $\mathrm{d}\mathcal{F}_{\mathbf{x}}$ the zero-average, unit variance Gaussian probability distributions. Note that in $\mathcal{F}_{\mathbf{x}}$, $\pi_1^* \mathrm{d}F = \frac{1}{(2\pi)^n} \mathrm{d}F_{\mathbf{x}}$. The coarea formula for $(\mathcal{V}, M, \pi_2, F)$ (Theorem 2.9) is

$$\mathbb{E}(\#(Z(\mathbf{f}) \cap \mathcal{K})) = \frac{1}{(2\pi)^n} \int_{\mathcal{K}} \mathrm{d}M(\mathbf{x}) \int_{F_{\mathbf{x}}} NJ(\mathbf{f}, i\mathbf{x})^{-2} \mathrm{d}F_{\mathbf{x}}$$

with Normal Jacobian $NJ(\mathbf{f},\mathbf{x}) = \det(D\pi_2(\mathbf{f},\mathbf{x})D\pi_2(\mathbf{f},\mathbf{x})^*)^{1/2}$.

The Normal Jacobian can be computed by

$$\begin{aligned}
NJ(\mathbf{f},\mathbf{x})^2 &= \det\left(D\mathbf{f}(\mathbf{x})^{-*} \begin{bmatrix} K_1(\mathbf{x},\mathbf{x}) & & \\ & \ddots & \\ & & K_n(\mathbf{x},\mathbf{x}) \end{bmatrix} D\mathbf{f}(\mathbf{x})^{-1} \right) \\
&= \frac{\prod K_i(\mathbf{x},\mathbf{x})}{|\det D\mathbf{f}(\mathbf{x})|^2}
\end{aligned}$$

We pick an arbitrary system of coordinates around $\mathbf{x}$. Using Lemma 2.3,

$$|\det D\mathbf{f}(\mathbf{x})|^2 \mathrm{d}M = \bigwedge_{i=1}^{n} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} f_i(\mathbf{x}) \overline{\frac{\partial}{\partial x_k} f_i(\mathbf{x})}\ \frac{\sqrt{-1}}{2} \mathrm{d}x_j \wedge d\bar{x}_k$$

Thus,

$$\begin{aligned}
&\mathbb{E}(\#(Z(\mathbf{f}) \cap \mathcal{K})) = \\
&\quad = \frac{1}{(2\pi)^n} \int_{\mathcal{K}} \bigwedge_{i=1}^{n} \sum_{jk} \int_{\mathcal{F}_{ix}} \frac{\langle Df(x)\frac{\partial}{\partial x_j}, Df(\mathbf{x})\frac{\partial}{\partial x_k}\rangle}{K_i(\mathbf{x},\mathbf{x})} \\
&\qquad\qquad \frac{\sqrt{-1}}{2} \mathrm{d}x_j \wedge \mathrm{d}\bar{x}_k \ \mathrm{d}\mathcal{F}_{i\mathbf{x}}(f_i) \\
&\quad = \frac{1}{\pi^n} \int_{\mathcal{K}} \bigwedge_{i=1}^{n} \sum_{jk} \omega_i \left(\frac{\partial}{\partial x_j}, J\frac{\partial}{\partial x_k} \right) \frac{\sqrt{-1}}{2} \mathrm{d}x_j \wedge \mathrm{d}\bar{x}_k \\
&\quad = \frac{1}{\pi^n} \int_{\mathcal{K}} \bigwedge_{i=1}^{n} \omega_i(\mathbf{x})
\end{aligned}$$

using Proposition 3.9. □

13. Affine and multi-homogeneous setting

We start by particularizing Theorem 3.11 for the Bézout Theorem setting.

The space $\mathcal{P}_{d_i}$ of all polynomials of degree $\leq d_i$ is endowed with the Weyl inner product [13] given by

$$\langle \mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \rangle = \begin{cases} \binom{d_i}{\mathbf{a}}^{-1} & \text{if } \mathbf{a} = \mathbf{b} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

With this choice, $\mathcal{P}_{d_i}$ is a non-degenerate fewspace with Kernel

$$K(\mathbf{x}, \mathbf{y}) = \sum_{|\mathbf{a}| \leq d_i} \binom{d_i}{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \bar{\mathbf{y}}^{\mathbf{a}} = (1 + \langle \mathbf{x}, \mathbf{y} \rangle)^{d_i}$$

The geometric reason behind Weyl's inner product will be explained in the next section. A consequence of this choice is that the metric depends linearly in d_i.

We compute $K_{j\cdot}(\mathbf{x}, \mathbf{x}) = d_j \bar{x}_j K(\mathbf{x}, \mathbf{x})/R^2$ and

$$K_{jk}(\mathbf{x}, \mathbf{x}) = \delta_{jk} d_i K(\mathbf{x}, \mathbf{x})/R^2 + d_i(d_{i-1}) \bar{x}_j x_k / R^4,$$

with $R^2 = 1 + \|\mathbf{x}\|^2$. Lemma 3.10 implies

$$g_{jk} = d_i \left(\frac{1}{R^2} \left(\delta_{jk} - \frac{\bar{x}_j x_k}{R^2} \right) \right),$$

with $R^2 = 1 + \|\mathbf{x}\|^2$. Thus, if ω_i is the metric form of $\mathcal{P}_{d_i}$ and ω_0 the metric form of $\mathcal{P}_1$,

$$\bigwedge_{i=1}^{n} \omega_1 = (\prod_{i=1}^{n} d_i) \bigwedge_{i=1}^{n} \omega_0.$$

Comparing the bounds in Theorem 3.11 for the linear case (degree 1 for all equations) and for $\mathbf{d}$, we obtain:

Corollary 3.12. *Let* $\mathbf{f} \in \mathcal{P}_{\mathbf{d}} = \mathcal{P}_{d_1} \times \cdots \times \mathcal{P}_{d_n}$ *be a zero average, unit variance variable. Then,*

$$\mathbb{E}(n_{\mathbb{C}^n}(\mathbf{f})) = \prod d_i$$

Remark 3.13. Mario Wschebor pointed out that if one could give a similar expression for the variance (which is zero) it would be possible to deduce and 'almost everywhere' Bézout's theorem from a purely probabilistic argument.

Now, let $\mathcal{F}_i$ is the space of polynomials with degree d_{ij} in the j-th set of variables. We write $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_s)$ for $\mathbf{x}_i \in \mathbb{C}^{n_i}$, and the same convention holds for multi-indices.

The inner product will be defined by:

$$\langle \mathbf{x}_1^{\mathbf{a}_1} \ldots \mathbf{x}_s^{\mathbf{a}_n}, \mathbf{x}_1^{\mathbf{b}_1} \ldots \mathbf{x}_s^{\mathbf{b}_n} \rangle = \frac{\delta_{\mathbf{a}_1\mathbf{b}_1} \cdots \delta_{\mathbf{a}_s\mathbf{b}_s}}{\binom{d_{i1}}{\mathbf{a}_1} \cdots \binom{d_{is}}{\mathbf{a}_s}} \tag{8}$$

The integral kernel is now

$$K(\mathbf{x}, \mathbf{y}) = (1 + \langle \mathbf{x}_1, \mathbf{y}_1 \rangle)^{d_{i1}} \cdots (1 + \langle \mathbf{x}_s, \mathbf{y}_s \rangle)^{d_{is}}$$

We need more notations: the j-th variable belongs to the $l(j)$-th group, and $R_l^2 = 1 + \|\mathbf{x}_l\|^2$.

With this notations,

$$\begin{aligned}
K_{j\cdot}(\mathbf{x}, \mathbf{x}) &= d_{l(j)} \frac{\bar{\mathbf{x}}_j K(\mathbf{x}, \mathbf{x})}{R_{l(j)}^2} \\
K_{jk}(\mathbf{x}, \mathbf{x}) &= \delta_{jk} d_{l(j)} \frac{K(\mathbf{x}, \mathbf{x})}{R_{l(j)}^2} + d_{l(j)}(d_{l(k)} - \delta_{l(j)l(k)}) \frac{\bar{\mathbf{x}}_j \mathbf{x}_k}{R_{l(j)}^2 R_{l(k)}^2} \\
g_{jk} &= d_{l(j)} \left(\frac{\delta_{jk}}{R_{l(j)}^2} - \delta_{l(j)l(k)} \frac{\bar{\mathbf{x}}_j \mathbf{x}_k}{R_{l(j)}^2 R_{l(k)}^2} \right)
\end{aligned}$$

Recall that ω_i is the symplectic form associated to $\mathcal{F}_i$. We denote by ω_{jd} the form associated to the polynomials that have degree $\leq d$ in the j-th group of variables, and are independent of the other variables. From the calculations above,

$$\omega_i = \omega_{1d_1} + \cdots + \omega_{sd_s} = d_{i1}\omega_{11} + \cdots + d_{is}\omega_{s1}$$

Hence,

$$\bigwedge \omega_i = \bigwedge d_{i1}\omega_{11} + \cdots + d_{is}\omega_{s1}.$$

This is a polynomial in variables $Z_1 = \omega_{11}, \ldots, Z_s = \omega_{ss}$. Notice that $Z_1 \wedge Z_2 = Z_2 \wedge Z_1$ so we may drop the wedge notation. Moreover, $Z_i^{n_i+1} = 0$. Hence, only the monomial in $Z_1^{n_1} Z_2^{n_2} \cdots Z_s^{n_s}$ may be nonzero.

Corollary 3.14. *Let B be the coefficient of $Z_1^{n_1} Z_2^{n_2} \cdots Z_s^{n_s}$ in*

$$\prod (d_{i1} Z_1 + \cdots + d_{is} Z_s).$$

Let $\mathbf{f} \in \mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$ be a zero average, unit variance variable. Then,

$$\mathbb{E}(n_{\mathbb{C}^n}(\mathbf{f})) = B$$

Proof. By Theorem 3.11,

$$\begin{aligned} \mathbb{E}(n_{\mathbb{C}^n}(\mathbf{f})) &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \bigwedge \omega_i \\ &= \frac{B}{\pi^n} \int_{\mathcal{K}} \underbrace{\omega_{11} \wedge \cdots \wedge \omega_{11}}_{n_1 \text{times}} \wedge \cdots \wedge \underbrace{\omega_{s1} \wedge \cdots \wedge \omega_{s1}}_{n_s \text{times}} \end{aligned}$$

In order to evaluate the right-hand-term, let $\mathcal{G}_j$ be the space of affine polynomials on the j-th set of variables. Its associated symplectic form is ω_{i1}.

A generic polynomial system in

$$\mathcal{G} = \underbrace{\mathcal{G}_1 \times \cdots \mathcal{G}_1}_{n_1 \text{times}} \times \cdots \times \underbrace{\mathcal{G}_s \times \cdots \mathcal{G}_s}_{n_s \text{times}}$$

is just a set of decoupled linear systems, hence has one root. Hence,

$$1 = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \underbrace{\omega_{11} \wedge \cdots \wedge \omega_{11}}_{n_1 \text{times}} \wedge \cdots \wedge \underbrace{\omega_{s1} \wedge \cdots \wedge \omega_{s1}}_{n_s \text{times}}$$

and the expected number of roots of a multi-homogeneous system is B. $\square$

Exercise 3.3. The Frobenius norm for tensors $T_{j_1 \cdots j_q}^{i_1 \cdots i_p}$ is

$$\|T\|_F = \sqrt{\sum_{i_1, \cdots, j_q = 1}^{n} |T_{j_1 \cdots j_q}^{i_1 \cdots i_p}|^2}$$

The unitary group acts on the variable j_1 by composition:

$$T_{j_1 \cdots j_q}^{i_1 \cdots i_p} \overset{U}{\rightsquigarrow} \sum_{k=1}^{N} T_{k \cdots j_q}^{i_1 \cdots i_p} U_{j_1}^{k}.$$

Show that the Frobenius norm is invariant for the $U(n)$-action. Deduce that it is invariant when $U(n)$ acts simultaneously on all lower (or upper) indices. Deduce that Weyl's norm is invariant by unitary action $f \rightsquigarrow f \circ U$.

Exercise 3.4. This is another proof that the inner product defined in (7) is $U(n+1)$-invariant. Show that for all $f \in \mathcal{H}_d$,

$$\|f\|^2 = \frac{1}{2^d d!} \int_{\mathbb{C}^{n+1}} \|f(\mathbf{x})\|^2 \, \frac{1}{(2\pi)^{n+1}} e^{-\|\mathbf{x}\|^2/2} \, \mathrm{d}V(\mathbf{x}).$$

The integral is the $\mathcal{L}^2$ norm of $\mathbf{f}$ with respect to zero average, unit variance probability measure. Conclude that $\|f\|$ is invariant.

Exercise 3.5. Show that if $\mathcal{F} = \mathcal{H}_d$, then the induced norm defined in Lemma 3.10 is d times the Fubini-Study metric. Hint: assume without loss of generality that $\mathbf{x} = \mathrm{e}_0$.

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RESÚMENES DE CONFERENCIAS

1. Conferencias plenarias

Mario Wschebor (por Enrique Cabaña)

Algunos apuntes sobre su trayectoria científica y humana

Entropy rigidity of non-positively curved symmetric spaces (por François Ledrappier)

We characterize locally symmetric compact spaces of nonpositive curvature by the equality of general inequalities between asymptotic geometric quantities.

De la trampa al coalescente: variación genética en especies de ratones de la Patagonia (por Enrique Lessa)

Se presentará un estudio en marcha orientado a detectar señales de la historia de las poblaciones, en particular en relación a los ciclos climáticos del cuaternario ("la era del hielo"), en colecciones de secuencias de ADN. Se presentará una introducción al modelo coalescente estándar y a modelos estructurados en el espacio, y resultados de su aplicación al caso de estudio.

Matemática Reversa (por Antonio Montalbán)

La matemática reversa es un área de la lógica cuyo objetivo es determinar qué axiomas son realmente necesarios para las distintas áreas de la matemática. Esta área de la lógica esta estrechamente conectada con la matemática computable, que trata de medir la complejidad de los objetos matemáticos en términos computacionales. Describiremos las principales ideas de esta área de la lógica.

La geometría de contacto del problema de los 3 cuerpos (por Gabriel Paternain)

Sabemos desde hace mucho tiempo la forma de escribir las ecuaciones de movimiento de un satélite que se mueve bajo la influencia de los campos gravitatorios de la Tierra y la Luna, pero, sorprendentemente, todavía no entendemos completamente el comportamiento a largo plazo del satélite ya que no hay forma explícita de n resolver las ecuaciones. A finales del siglo 19, Poincaré notó la presencia de caos en el sistema y así propició el nacimiento de la teoría moderna de sistemas dinámicos.Recientemente, un nuevo tipo de geometría llamada geometría de contacto (el pariente de dimensión impar de la geometría simpléctica) se ha propuesto como herramienta para la comprensión de este viejo problema de la mecánica celeste. En la charla voy a tratar de explicar lo que es la geometría de contacto y por qué es relevante para el problema de 3 cuerpos.

Solving systems of polynomial equations: some recent results (par Michael Shub)

I will survey some recent results with coworkers, Diego Armentano, Carlos Betran, Jean-Pierre Dedieu and Gregorio Malajovich. In Complexity of Bezout's Theorem VI, I asserted the existence af an algorithm to approximate a homotopy path of (problem,solution) pairs whose number of steps is bounded by the length of the path in the condition metric. With Dedieu and Malajovich we present a specific algorithm which satisfies the same bounds and which takes into account computational errors. With Armentano we study the algorithm initially proposed by Smale in one variable in 1981 in the light of what we have learned in the intervening 30 years. New interesting problems arise.

2. Sesiones especiales

2.1 Álgebra y Geometría Algebraica:

Producto de categorías abelianas de Deligne
(Por Ignacio Lopez Franco)

En su articulo sobre categorías Tanakianas Deligne define y utiliza un producto tensorial de categorías abelianas. Esta construcción ha sido utilizada por varios autores en el estudio de categorías tensoriales, en conexión con la teoría de álgebras de Hopf. En esta exposición comparamos el producto de Deligne con un producto de categorías con colímites finitos que lo precede, explicamos las limitaciones de la definición de Deligne y como solucionarlas.

Monoides algebraicos: su estructura y su geometría
(por Alvaro Rittatore)

En esta charla mostraremos los resultados iniciales en cuanto a la estructura y geometría de los monoides algebraicos y cómo se relacionan entre sí.

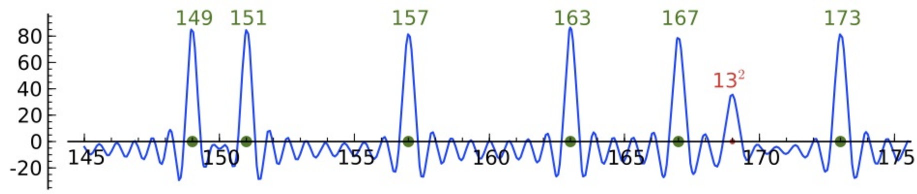
La charla está dirigida a estudiantes de la licenciatura en matemática, y asumirá muy pocos conocimientos previos (algebra lineal, anillos y módulos).

La armonía de los números primos - La hipótesis de Riemann
Gonzalo Tornaría

Hilbert, en su famosa presentación en el ICM de Paris de 1900, incluyó la hipótesis de Riemann como uno de los 23 problemas con los que desafió a los matemáticos del siglo XX. Cien años después, el Instituto Clay de Matemáticas lo coloca como uno de los 7 problemas del milenio, ofreciendo un premio de un millón de dólares.

En esta charla haremos una introducción histórica al problema de la distribución de los números primos, su relación con los ceros de la función zeta de Riemann, y la hipótesis de Riemann.

> *Hay dos hechos acerca de la distribución de los números primos de los cuales espero convencerlos de una manera tan contundente que quedarán permanentemente grabados en sus corazones. El primero es que, a pesar de su definición simple y su papel como componentes básicos de los números naturales, los números primos están entre los objetos más arbitrarios y vulgares estudiados por los matemáticos: crecen como malas hierbas entre los números naturales, pareciendo obedecer ninguna otra ley que la del azar, y nadie puede predecir dónde brotará el siguiente. El segundo hecho es aún más sorprendente, ya que expresa todo lo contrario: que los números primos exhiben una regularidad impresionante, que hay leyes que rigen su comportamiento, y que cumplen estas leyes con precisión casi militar.*
>
> *Don Zagier (1975)*



$$-\sum_{i=1}^{1000} \cos(\gamma_i \log x)$$

Criptografía y curvas elípticas
(por Soledad Villar)

La criptografía estudia conceptos y técnicas matemáticas relacionadas con la seguridad de la información. En esta charla voy a explicar las ideas matemáticas detrás de los sistemas criptográficos, principalmente los de clave pública. Luego comentaré a modo de ejemplo alguno de los problemas en los que se basa la criptografía de clave pública, en particular el problema del logaritmo discreto en curvas elípticas.

2.2 Análisis:

Dualidad para productos cruzados por acciones parciales (por Fernando Abadie)

En el marco del estudio de la estructura de los factores de tipo III, M. Takesaki publicó en 1973 el primer resultado sobre dualidad de productos cruzados de álgebras de von Neumann por grupos abelianos localmente compactos. A partir de entonces se fueron sucediendo, de manera intensa, estudios similares para productos cruzados de álgebras en diversas categorías y para acciones de variados objetos. En nuestra exposición propondremos una forma de la dualidad que consideramos es la adecuada para el caso de acciones parciales de grupos en C*-álgebras. Como aplicación mostraremos cómo es posible globalizar siempre tales acciones parciales.

2.3 Probabilidad y Estadística:

Recomendación colaborativa (por Ricardo Fraiman)

Recomendación colaborativa (CR) es una herramienta a usar en la WEB que típicamente recoje información sobre tus intereses personales y compara tu perfi

l con otros usuarios con gustos similares. El objetivo del sistema es dar recomendaciones personalizadas a los usuarios.

2.4 Sistemas Dinámicos y Geometría diferencial:

Trayectorias periódicas en billares triangulares (por Alfonso Artigue)

Considere un triángulo y una masa o bola puntual en su interior. El movimiento de la bola en el interior del triángulo es rectilíneo uniforme, podemos suponer que la velocidad tiene norma 1 en todo momento. Cuando la bola llega a un borde del triángulo, ésta cambia instantáneamente de dirección según la regla del espejo "ángulo de incidencia = ángulo de reflexión". En general si la bola llega a un vértice del triángulo, consideramos que el movimiento se detiene. El problema es el siguiente: ¿existe en todo triángulo por lo menos una trayectoria periódica? De otra manera: ¿existe en todo triángulo una posición inicial y una velocidad inicial tales que luego de una cantidad finita de rebotes la bola llegue al punto del que partió con la misma velocidad? Este problema aún no ha sido resuelto. En la charla mostraremos algunas soluciones conocidas y posibles técnicas que lleven a su solución.

Deformaciones casi-simétricas de espacios métricos compactos (por Matías Carrasco)

Los homeomorfismos casi-simétricos son una generalización de los casi-conformes a espacios métricos en general, y aparecen de forma natural en geometría hyperbólica y dinámica compleja. El estudio de invariantes de casi-simetría, como la dimensión conforme, tiene aplicaciones importantes en teoría geométrica de grupos.

En la charla definiré estas nociones y mostraré como la dimensión conforme de un espacio métrico compacto está fuertemente relacionada con módulos combinatorios de familias de curvas en dicho espacio. Esto permite dar un criterio general de dimensión conforme uno, y responder parcialmente a la pregunta: cuáles son los grupos hyperbólicos cuyo borde tiene dimensión conforme uno?

Clasificación de conjuntos minimales de homeomorfismos del toro (por Alejandro Passeggi)

En sistemas dinámicos existe una generalización muy natural de órbita periódica que es lo que se llama conjunto minimal. Mas aún, asumiendo la compacidad del espacio de fase se garantiza la existencia de dichos conjuntos cualquiera sea el sistema, por lo cual se pueden considerar conjuntos básicos en la teoría. En la charla se expondrá una clasificación topológica de los conjuntos minimales para homeomorfismos del toro $\mathbb{T}^2$, que surge de un trabajo conjunto con Tobias Jäger y Ferry Kwakkel. Por último se presentarán algunas relaciones interesantes entre la teoría de rotación y la clasificación dada.

Dinámica genérica en superficies: Existencia de atractores hiperbólicos (por Rafael Potrie)

La dinámica C^1-genérica se preocupa de estudiar propiedades dinámicas de difeomorfismos "típicos"de una variedad. Por típicos se entiende que verifican propiedades que son abundantes en la topología en cuestión, en este caso la topología C^1.

Voy a intentar dar un pantallazo (lo más autocontenido posible) sobre este tema. Me voy a concentrar en el caso de superficies que es el contexto más sencillo. Intentaré también presentar algunas ideas que permiten probar que para un difeomorfismo C^1-genérico de una superficie existe un atractor hiperbólico (esto es parte de la tesis de doctorado de A. Araujo, un alumno de Mañe, pero debido a que contenía un error en su prueba, esto nunca fue publicado y paso a ser parte del Folklore, mi prueba se basa en técnicas recientes de dinámica C^1-genérica).

Charla a confirmar
(por Andrés Sambarino)

Érase una vez el caos
(por Martín Sambarino)

Partial hyperbolicity and ergodicity in dimension 3
(por Raúl Ures)

A dynamical system is partially hyperbolic if it has three invariant directions E^s, E^c and E^u, being E^s uniformly contracting, E^u uniformly expanding while E^c has an intermediate behavior. The study of partially hyperbolic systems has been one of the most active topics in dynamics in the last two decades. The purpose of this talk will be to present the state of the art in the study of the ergodicity of conservative partially hyperbolic diffeomorphisms on three dimensional manifolds.

In a previous work (joint with Jana and Federico Rodriguez Hertz) we proved the Pugh-Shub conjecture for partially hyperbolic diffeomorphisms with 1-dimensional center, i.e. stably ergodic diffeomorphisms are dense among the partially hyperbolic ones. In subsequent results, we obtained, jointly with the same co-authors, a more accurate description of this abundance of ergodicity in dimension three. We will describe these results, some recent advances and the main open problems and conjectures on the subject.

2.5 Matemática Aplicada y otras ciencias:

Biología de Sistemas. Nuevos enfoques para abordar problemas desafiantes de la Biología (por Luis Acerenza)

Los seres vivos más simples presentan una gran complejidad, originada en el alto número de componentes moleculares y en las características de sus interacciones. La mayor parte de lo que conocemos de estos seres vivos es el fruto de la aplicación de las metodologías reduccionistas. Sin embargo, existe la percepción de que muchos problemas de la Biotecnología y la Biomedicina, que siguen hoy sin resolverse en forma satisfactoria, requieren del empleo de estrategias sistémicas. Como respuesta a esta percepción, hace aproximadamente una década, se funda la Biología de Sistemas. Es una nueva aproximación, de carácter netamente interdisciplinario, que reúne a Matemáticos, Físicos, Químicos, Informáticos e Ingenieros, además de investigadores de variadas áreas de la Biología, entre otros, con el fin de dar respuesta a algunas interrogantes particularmente desafiantes de la Biología. En la exposición se mencionarán algunas de las metodologías sistémicas que están siendo desarrolladas y los tópicos de Matemática involucrados.

Charla a confirmar (por Héctor Cancela)

Simulación de secuencias sobre alfabetos finitos (por Alvaro Martin)

Proponemos un esquema de simulación para secuencias de símbolos sobre un alfabeto finito, en el cual se genera aleatoriamente una secuencia “simulada”, y, a partir de una de entrenamiento, x. La muestra de entrenamiento se trata como una secuencia individual; no se asume ningún modelo probabilístico sobre ella. Se muestra que la simulación y, generada de este modo, es estadísticamente similar a la muestra de entrenamiento x, en un sentido a ser formalizado durante la presentación. Al mismo tiempo se maximiza, esencialmente, la incertidumbre de y dado x, bajo la restricción de que sean estadísticamente similares. Esta técnica tiene aplicaciones, por ejemplo, en la simulación de texturas de imágenes, ruido de audio, secuencias de eventos en ambientes simulados, entre otras.

Optimización de frecuencias en sistemas de transporte público (por Antonio Mautone)

El problema de la determinación de frecuencias en sistemas de transporte público consiste en definir el intervalo de tiempo entre sucesivos vehículos para un conjunto dado de líneas. Este problema surge ya sea en la planificación estratégica (como sub-problema de la definición de recorridos) como en la táctica (por ejemplo, para ajustar los servicios en una estación del año u hora del día en particular). Métodos de optimización pueden ser aplicados en este contexto, donde el interés de los usuarios (tiempo de viaje, nivel de ocupación de los vehículos) y operadores (costos de operación) deben tenerse en cuenta, asícomo restricciones impuestas por la infraestructura (capacidad de las calles y de los vehículos).

En particular, consideramos el problema de la determinación de frecuencias como aquel que minimiza el tiempo total de viaje de los usuarios, sujeto a una restricción en el tamaño de la flota de vehículos. Formulamos el problema de optimización de frecuencias como un programa no lineal, el cual es posteriormente linealizado mediante una discretización del dominio de las frecuencias. La formulación resultante es de tipo lineal entera mixta. La resolución de este problema conlleva atacar un problema de optimización combinatoria, que es difícil de resolver en forma exacta para instancias de gran tamaño (en el sentido de la cantidad de líneas del sistema). Por ese motivo, se propone un método aproximado de resolución, basado en la metaheurística Tabu Search.

Se presentan resultados numéricos de la aplicación del modelo exacto y de la metaheurística sobre diferentes casos de prueba. Asimismo se presenta una discusión de las implicancias sobre la estructura del modelo matemático, de la inclusión de restricciones que representan decisiones del planificador, que deben tener en cuenta las decisiones de los usuarios (formulación en dos niveles).

Estimación subpixelica de la Point Spread Function de una cámara digital en presencia de aliasing (por Pablo Musé)

Most medium to high quality digital cameras (DSLRs) acquire images at a spatial rate which is several times below the ideal Nyquist rate. For this reason only aliased versions of the cameral point-spread function (PSF) can be directly observed. Yet, it can be recovered, at a sub-pixel resolution, by a numerical method. Since the acquisition system is only locally stationary, this PSF estimation must be local.

In this talk we present a theoretical study proving that the subpixel PSF estimation problem is well-posed even with a single well chosen observation. Indeed, theoretical bounds show that a near-optimal accuracy can be achieved with a calibration pattern mimicking a Bernoulli(0.5) random noise. The physical realization of this PSF estimation method is demonstrated in many comparative experiments. The PSF estimates reach stringent accuracy levels. To the best of our knowledge, such a regularization-free and model-free subpixel PSF estimation scheme is the first of its kind.

The optimal harvesting problem under price uncertainty (por Adriana Piazza)

In this seminar we will present the exploitation of a one species forest plantation when timber price is governed by a time stochastic process. The work focuses on providing closed expressions for the optimal harvesting policy in terms of the parameters of the price process and the discount factor. We assume that harvest is restricted to mature trees older than a certain age and that growth and natural mortality after maturity are neglected.

We will start the talk with a brief review of the relevant results in the deterministic case:

- Identical trees (Faustmann 1849 [1])
- Age-class model (Rapaport et al. 2003 [3])

After that, we will present the results obtained in the stochastic framework [2]. We use stochastic dynamic programming techniques to characterize the optimal policy for two important cases: (i) when prices follow a geometric Brownian motion we completely characterize the optimal policy for all possible choices of drift and discount factor, (ii) if prices are governed by a mean-reverting (Ornstein-Uhlenbeck) process we provide sufficient conditions, based on explicit expressions for reservation prices, assuring that harvesting everything available is optimal. For the Ornstein-Uhlenbeck process, we propose a policy based on a reservation price that perfoms well in numerical simulations. In both cases we solve the problem for *every* initial condition and the best policy is obtained endogenously, that is, without imposing any ad hoc restrictions such as maximum sustained yield or convergence to a predefined final state.

Following [5], we consider a nested formulation based on conditional risk mappings, a concept defined in [4]:

$$V_0(x, p(0)) =$$

$$\min_{c(0)} \left\{ -p(0)c(0) + \delta\rho_{1|p(0)} \left[\min_{c(1)} -p(1)c(1) + \ldots + \delta\rho_{T|p(T-1)} \left[\min_{c(T)} -p(T)c(T) \right] \right] \right\}.$$

with the corresponding dynamic programming equations,

$$V_t(x(t), p(t)) = \min_{c(t)} \Big\{ -p(t)c(t) + \delta\rho_{t+1|p(t)} \big(V_{t+1}(x(t+1), p(t)) \big) \Big\}$$

$$t = 0, \ldots, T-1.$$

We consider the risk neutral framework, where $\rho = \mathbb{E}$ and the risk averse case with the risk measure $\rho = \mathrm{CVaR}_\alpha$.

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La ciencia de la Variabilidad climática y el Cambio Climático (por Madeleine Renon)

La variabilidad climática y el cambio climático, son tópicos muy utilizados hoy en día, ¿ pero cuanto sabemos en realidad ? ¿ Qué se puede decir del punto del vista científico?, ¿ cuales son las herramientas que se utilizan? y lo mas importante, ¿ que es lo que no se sabe aun ? Si bien el clima de una región se define generalmente por variables atmosféricas, debemos saber que ellas responden a variaciones de un sistema más complejo: El sistema climático. El sistema climático esta compuesto por la atmósfera, los océanos, la criosfera, la biosfera y los continentes; cada uno de ellos con diferentes tiempos de respuesta a las forzantes pudiendo presentar respuestas lineales y no lineales, estas características hacen que el sistema climático sea difícil de simular numéricamente. El sistema por otro lado presenta diferentes variabilidades temporales naturales, desde la variabilidad interanual pasando a la decadal o multi-decadal, hasta el cambio climático. Esta charla intentara presentar al público las definiciones básicas sobre el tema, las herramientas científicas que se utilizan y los desafíos que se nos presentan.

Un Sistema eficiente de intercambio de video en vivo sobre Internet (por Pablo Romero)

Ya desde el Siglo III se conocían técnicas para engañar el ojo humano con movimientos rápidos, elemento básico para generar video. Sin embargo, la generación y distribución masiva de video ha sido posible recién a fines del Siglo XX. Revisaremos los desafíos inherentes a la distribución exitosa de video sobre Internet. El uso eficiente de recursos motiva el despliegue de Redes de Pares (o redes P2P). Existe una importante diversidad de modelos matemáticos que procuran comprender el funcionamiento de estas redes, que prometen entrega de video en vivo de alta calidad. Analizaremos un modelo matemático que se concentra en la cooperación de los usuarios, que intercambian fragmentos de video. Se introduce una nueva técnica de intercambio que mejora sistemas ampliamente utilizados en Internet, como lo es BitTorrent.

3. Educación Matemática y Matemática Elemental:

Los 5 sólidos platónicos y la teoría de grafos (por Eduardo Canale)

La existencia de sólo cinco poliedros convexos cuyas caras son polígonos regulares, conocidos como sólidos platónicos, admite una demostración trivial usando la restricción en los ángulos de las caras. Sin embargo, su existencia trasciende la geometría métrica y podemos demostrar que aunque las aristas y las caras fueran curvas y superficies alabeadas, el solo hecho de ser regulares es la razón por la cual son solo 5. En esta charla daremos la demostración de éste hecho usando teoría de grafos.

Teoría de la Aritmética Transfinita de George Cantor (por Enrique Espínola)

La actividad tendrá como columna vertebral, las siguientes etapas:

1. ¿Por qué una charla sobre temas no curriculares de Matemática?
2. ¿Por qué sobre George Cantor?
3. ¿Por qué sobre la obra de Cantor?
4. Aspectos básicos de la Teoría Aritmética Transfinita.

Se supone por parte de los participantes, conocimientos básicos sobre la Teoría de Conjuntos, vista en cursos de Educación Media.

1. ¿Cuándo se puede decir que un conjunto es infinito?
2. Se considerarán diversas Paradojas como las de Russell: "Un barbero afeita solo a aquellas personas que no se afeitan a sí mismas, entonces ¿Quién afeita al barbero?"
3. Se describirá la metáfora conocida como "El Hotel de Hilbert"
4. Realmente ¿existe la misma cantidad de Naturales que Naturales Pares? ¿Existe la misma cantidad de puntos en una recta que en un plano o en el espacio tri dimensional?
5. ¿Qué cosa es un Alef? ¿Es todo número transfinito un Alef? ¿Los números Cardinales y los números Ordinales son iguales, es decir coinciden en todo conjunto?

Estas y otras muchas preguntas, intentarán hallar respuestas en esta charla introductoria, así como otras preguntas, habrá que intentar contestar, por el simple hecho de que aun, para estas, no existe respuesta alguna. Para estudiantes de nivel medio y terciario, Teoría de la Aritmética Transfinita de George Ferdinand Ludwing Philipp Cantor.

Análisis de frecuencias: del arcoiris al sonido digital (por Omar Gil)

La luz del sol puede descomponerse en el continuo de colores que vemos en el arcoris, y el sonido de una cuerda de guitarra describirse como la superposición de una nota fundamental y sus infinitos armónicos. Pero el sonido digital es infinitamente más simple, porque en cada intervalo de tiempo hay una cantidad finita de tomas de sonido, y el análisis requiere sólo una cantidad finita de frecuencias. Por lo tanto, todo se reduce a cálculo con matrices y a una geometría que es como la del plano y el espacio, aunque con más dimensiones.

En esta charla haré una presentación de aspectos básicos de esta área, involucrados en cosas tan cotidianas como un reproductor de mp3 o la representación de una imagen en el formato jpg, enfatizando las conexiones con ideas matemáticas que los profesores trabajan durante la educación media.

Análisis del discurso como acción social: su rol en la construcción y difusión de conocimiento matemático (por Verónica Molfino)

¿ Por qué el concepto de límite se enseña, en el contexto educativo uruguayo, de la manera en que se hace? Una manera de responder esta pregunta sería considerando el marco de la Transposición Didáctica (Chevallard). Pero hemos descubierto que no es suficiente analizar cómo el conocimiento se transforma de un saber sabio a un saber a enseñar... para responder esa pregunta entendemos necesario identificar los actores involucrados en todo un proceso que explica la transposición del ámbito científico al escolar, y en particular analizar sus prácticas. Presentamos pues un marco que permite problematizar el discurso matemático escolar (dme), a partir de la consideración del discurso como una acción social. En esta conferencia describiremos el dme y explicaremos, en particular para el concepto de límite y en el contexto educativo uruguayo, qué rol cumple en la construcción y difusión de conocimiento matemático.

Caminatas poligonales en el plano hiperbólico (por Pablo Lessa)

Explicaremos brevemente el modelo del disco de Poincaré para el plano hiperbólico. Compararemos las curvas descritas por un mismo juego de instrucciones en dicho modelo y en el plano Euclideo. Finalizaremos discutiendo el comportamiento de ciertas caminatas aleatorias en ambas geometrías.

Sistemas dinámicos en bachillerato (por Fernando Pelaez)

En esta charla compartiremos nuestra experiencia referida a la introducción, en el último año de bachillerato, de ideas y conceptos vinculados al área de la Matemática denominada "Sistemas Dinámicos". Además de la importancia intrínseca de los alcances de esta teoría, pensamos que las actividades desarrolladas permiten diversas estrategias didácticas y metodológicas, pues son ideales para trabajar simultáneamente en varios registros de representación (analíticos, geométricos, algebraicos y numéricos). Asimismo, la visualización es permanente, así como la utilización de la computadora, llegando incluso a descubrir algunas de sus eventuales limitaciones con la aparición de sistemas caóticos. Nuestra experiencia indica que este enfoque permite realizar una excelente síntesis de todo lo estudiado previamente en el curso.

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