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CENTRAL LIMIT THEOREM FOR THE NUMBER OF CROSSING OF RANDOM PROCESSES

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1. Introduction

This course presents the application of the Malevich [15],Cuzick [6] Berman [4] method for establishing a central limit theorem for non linear functional of Gaussian processes (see Section 3).These methods have been introduced in the 70's for studying zero crossing of stationary processes or the sojourn time of a stochastic process. We present here mainly its application to the number of roots of random processes. The basic argument is the approximation of the original process by a m-dependent process (see Section 3). Section 2 presents a short memento of crossings of process and the calculation of their moments. Our main tools and results are presented in Section 3. Section 4 presents generalizations and applications to some particular processes, in particular random trigonometric polynomials and specular point in sea-wave modeling.

2. Basic facts on crossings of functions

This section contains preliminary results almost without proofs. They can be found for example in Azaïs and Wschebor [3].

For simplicity all the functions $f(t)$ considered are real and of class C^1. If I is a real interval we will define:

$$N_u(f,I) := \#\{t \in I : \ f(t) = u\}.$$

$N_u(f,I)$, (N_u for short in case of no ambiguity) is the number of crossings of the level u or the number of roots of the equation $f(t) = u$ in the interval I. In a similar way, we define the number of up-crossings or down crossings:

$$U_u(f,I) := \#\{t \in I : \ f(t) = u, \ f'(t) > 0\}$$

$$D_u(f,I) := \#\{t \in I : \ f(t) = u, \ f'(t) < 0\}.$$

Down-crossings will not be considered in the sequel since the results are strictly equivalent to those for the up-crossings.

We will say that the real-valued function f defined on the interval $I = [t_1, t_2]$ satisfies hypothesis $H_{1.u}$ if:

- f is a function of class C^1;
- $f(t_1) \neq u$, $f(t_2) \neq u$;
- $\{t : t \in I, \ f(t) = u, \ f'(t) = 0\} = \emptyset$.

Key words and phrases. Rice formula,Wiener Chaos, Gaussian processes.

Proposition 1 (Kac's counting formula). *If f satisfies $H_{1.u}$, then*

$$N_u(f,I) = \lim_{\delta\to 0}\frac{1}{2\delta}\int_I \mathbb{1}_{\{|f(t)-u|<\delta\}}\ |f'(t)|\ dt. \tag{1}$$

The Kac counting formula has a weak version that will be useful

Proposition 2 (Banach formula). *Assume that f is only absolutely continuous. Then for any bounded Borel-measurable function $g:\mathbb{R}\to\mathbb{R}$, one has:*

$$\int_{-\infty}^{+\infty} N_u(f,I)\ g(u)\ du = \int_I |f'(t)|g(f(t))\ dt. \tag{2}$$

This formula is a version of the change of variable formula for non one-to-one functions.

From these formula we deduce by passage to the limit the Rice formula that gives the factorial moments of the number of (up-) crossings. For simplicity we limit to the Gaussian case and to the first two moments.

Theorem 3 (Gaussian Rice formula). *Let $\mathcal{X} = \{X(t) : t \in I\}$, I a compact interval of the real line, be a Gaussian process having $\mathcal{C}^1$-paths.*

- *Suppose that for every point $t \in I$ the variance of $X(t)$ does not vanish. Then*

$$\mathrm{E}\big(N_u\big) = \int_I \mathrm{E}\big(|X'(t)|\Big|X(t)=u\big)p_{X(t)}(u)dt, \tag{3}$$

and the expression above is finite.

- *Suppose that*

(4) *for every $s \neq t \in I$, the distribution of $\big(X(s),X(t)\big)$ does not degenerate .*

Then

$$\mathrm{E}\big(N_u(N_u-1)\big) = \int_{I^2} \mathrm{E}\big(|X'(s)||X'(t)|\Big|X(s)=X(t)=u\big)p_{X(s),X(t)}(u,u)dt, \tag{5}$$

and the expression above may be finite or infinite.

Remarks: We have the same kind of formulas for the up-crossings if we replace $|X'(t)|$ by the positive part $(X'(t))^+$.

In case of stationary processes, assuming that the process is centered with variance 1, (3) takes the simpler form

$$\mathrm{E}\big(N_u\big) = 2\mathrm{E}\big(U_u\big) = |I|\frac{\sqrt{2\lambda_2}}{\sqrt{\pi}}\phi(u),$$

where $\phi(.)$ is the standard normal density.

A very important issue is the finiteness of the second (factorial) moment. For stationary processes a necessary and sufficient condition (in addition to (4)) is given by the Geman condition: let $\Gamma(.)$ be the covariance of the process and define the function $\theta(.)$ by means of

$$\Gamma(\tau) := \mathrm{E}\big(X(t)X(t+\tau)\big) = 1 - \frac{\lambda_2\tau^2}{2} + \theta(\tau).$$

The Geman condition [5]is

$$\int \frac{\theta'(\tau)}{\tau^2}d\tau \text{ converges at } \tau = 0^+, \tag{6}$$

More precisely we have the bound

Proposition 4. *Let $X(t)$ be a stationary Gaussian process with* $\mathrm{E}(X(t)) = 0$, $\mathrm{Var}(X(t)) = 1$. *Let $\Gamma(.)$ be its covariance function, we assume that for every $\tau > 0$, $\Gamma(\tau) \neq \pm 1$ and the Geman condition. Let $U_u = U_u([0,T])$, then*

$$\begin{aligned}
&\mathrm{E}\big((U_u)(U_u - 1)\big)\\
&\quad = 2\int_0^T (T-\tau)\mathrm{E}\big(|X(0)||X'(\tau)|\big|X(0) = X(\tau) = u\big) \times p_{X(0),X(\tau)}(u,u)d\tau\\
&\quad \leq 2\int_0^T (T-\tau)\frac{\theta'(\tau)}{\tau^2}d\tau.
\end{aligned}$$

Remark that because of the Rolle theorem: $N_u \leq 2U_u + 1$, thus the proposition above also gives a bound for the variance of the number of crossings.

3. Central limit theorem for non-linear functionals

Our next main tool will be chaos expansion and Hermite polynomials. These polynomials are orthogonal polynomials for the Gaussian measure $\phi(x)dx$ where ϕ is the standard normal density. The nth Hermite polynomial H_n can be defined by means of the identity:

$$\exp(tx - t^2/2) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

We have for example $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$.

For F in $L^2(\phi(x)\ dx)$, F can be written as

$$F(x) = \sum_{n=0}^{\infty} a_n H_n(x),$$

with

$$a_n = \frac{1}{n!}\int_{-\infty}^{\infty} F(x)H_n(x)\phi(x)dx,$$

and the norm of F in $L^2(\phi(x)dx)$ satisfies

$$||F||_2^2 = \sum_{n=0}^{\infty} a_n^2 n!.$$

The Hermite rank of F is defined as the smallest n such that $a_n \neq 0$. For our purpose, we can assume that this rank greater or equal than 1.

A useful standard tool to perform computations with Hermite polynomials and Gaussian variables is Mehler's formula which we state with an extension (see León and Ortega, [13]).

Lemma 5 (Generalized Mehler's formula)**.** *(a) Let (X,Y) be a centered Gaussian vector* $\mathrm{E}(X^2) = \mathrm{E}(Y^2) = 1$ *and* $\rho = \mathrm{E}(XY)$. *Then,*

$$\mathrm{E}(H_j(X)H_k(Y)) = \delta_{j,k}\rho^j.$$

(b) Let (X_1, X_2, X_3, X_4) be a centered Gaussian vector with variance matrix

$$\Sigma = \begin{pmatrix} 1 & 0 & \rho_{13} & \rho_{14}\\ 0 & 1 & \rho_{23} & \rho_{24}\\ \rho_{13} & \rho_{23} & 1 & 0\\ \rho_{14} & \rho_{24} & 0 & 1 \end{pmatrix}$$

Then, if $r_1 + r_2 = r_3 + r_4$,

$$\mathrm{E}\big(H_{r_1}(X_1)H_{r_2}(X_2)H_{r_3}(X_3)H_{r_4}(X_4)\big) = \sum_{(d_1,d_2,d_3,d_4)\in Z} \frac{r_1!r_2!r_3!r_4!}{d_1!d_2!d_3!d_4!}\rho_{13}^{d_1}\rho_{14}^{d_2}\rho_{23}^{d_3}\rho_{24}^{d_4},$$

where Z *is the set of* d_i*'s satisfying:* $d_i \geq 0$*;*

$$d_1 + d_2 = r_1 \ ; \ d_3 + d_4 = r_2 \ ; \ d_1 + d_3 = r_3 \ ; \ d_2 + d_4 = r_4. \tag{7}$$

If $r_1 + r_2 \neq r_3 + r_4$ *the expectation is equal to zero.*
Notice that the four equations in (7) are not independent, and that the set Z *is finite and contains, in general, more than one 4-tuple.*

Wiener chaos. Let $L^2(\Omega, \mathcal{A}, \mathrm{P})$ be the space of square integrable variables generated by the process $X(t), t \in \mathbb{R}$. This Hilbert space is the orthogonal sum of the Wiener chaos of order p, $p = 0, \dots, n, \dots$: $\mathcal{H}_p$. $\mathcal{H}_p$ is defined as the closed linear subspace of $L^2(\Omega, \mathcal{A}, \mathrm{P})$ generated by the variables $H_p(X(t)), t \in \mathbb{R}$. In particular the space $\mathcal{H}_1$ is simply the Gaussian space associated to $X(t)$. A good reference on this subject is the Nualart book [16].

3.1. **A first central limit theorem.** Let $\mathcal{X} = \{X(t) : t \in \mathbb{R}\}$ be a centered real-valued stationary Gaussian process. Without loss of generality, we assume that $\mathrm{Var}(X(t)) = 1 \ \forall t \in \mathbb{R}$. We want to consider functionals having the form:

$$T_t := 1/t \int_0^t F(X(s))\, ds, \tag{8}$$

where F is some function in $L^2(\phi(x)dx)$.

Set $\mu := \mathrm{E}(F(Z))$, Z being a standard normal variable. μ is well defined. The Maruyama Theorem implies that if the spectral measure of the process $X(t)$ has no atoms, it is ergodic and T_t converges almost surely to μ. Our aim is to compute the speed of convergence and establish for it a central limit theorem.

For the statement of the next result, which is not hard to prove, we need the following additional definition.

Definition 6. *Let* m *be some positive real, the Gaussian process* $\{X(t) : t \in \mathbb{R}\}$ *is called "*m*-dependent" if* $\mathrm{Cov}(X(s), X(t)) = 0$ *whenever* $|t - s| > m$.

An example of such a 1-dependent process is the Slepian process which is stationary with covariance $\Gamma(t) = (1 - t)^+$.

Theorem 7 (Hoeffeding and Robins [7])**.** *With the notations and hypotheses above, if the process* $X(t)$ *is* m *dependent, then*

$$\sqrt{t}\bigg(1/t \int_0^t F(X(s)) - \mu\, ds\bigg) \to N(0, \sigma^2) \text{ in distribution as } t \to +\infty,$$

where

$$\sigma^2 = \frac{1}{m}\mathrm{Var}\Big(\int_0^m F(X(s))ds\Big).$$

The proof is easy by the "shortening method": we cut $[0, T]$ into smaller intervals separated by gaps of size m giving the independence.

Our aim is to extend this result to processes which are not $m-$dependent. The proof we present follows Berman [4] with a generalization, due to Kratz and León [10] , to functions F in (8) having an Hermite rank not necessarily equal to 1.

For $\varepsilon > 0$, we will approximate the given process $X(t)$ by a new one $X_\varepsilon(t)$ which is $1/\varepsilon$-dependent and estimate the error.

As an additional hypothesis, we will assume that the process $X(t)$ has a spectral density $f(\lambda)$. It has the following spectral representation:

$$X(t) = \sqrt{2}\int_0^\infty \left[\cos(t\lambda)\sqrt{f(\lambda)}dW_1(\lambda) + \sin(t\lambda)\sqrt{f(\lambda)}dW_2(\lambda)\right], \tag{9}$$

where W_1 and W_2 are two independent Wiener processes (Brownian motions). Indeed, using isometry properties of the stochastic integral, it is easy to see that the process given by (9) is centered, Gaussian and with the good covariance:

$$\begin{aligned}\Gamma(t) &= \mathrm{E}(X(s)X(s+t))\\ &= 2\int_0^\infty \cos(\lambda s)\cos(\lambda(t+s))f(\lambda)d\lambda + 2\int_0^\infty \sin(\lambda s)\sin(\lambda(t+s))f(\lambda)d\lambda\\ &= 2\int_0^\infty \cos(\lambda t)f(\lambda)d\lambda.\end{aligned}$$

Define now the function $\psi(.)$ as the convolution $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} * \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}$. This function is even, non negative, $\psi(0) = 1$, has support included in $[-1,1]$ and a non-negative Fourier transform. Set $\psi_\varepsilon(.) := \frac{1}{\varepsilon}\psi(\varepsilon .)$ and let $\widehat{\psi_\varepsilon}$ be its Fourier transform. Define

$$X^\varepsilon(t) := \sqrt{2}\int_0^\infty \left[\cos(t\lambda)\sqrt{f * \widehat{\psi_\varepsilon}(\lambda)}dW_1(\lambda) + \sin(t\lambda)\sqrt{f * \widehat{\psi_\varepsilon}(\lambda)}dW_2(\lambda)\right], \tag{10}$$

where the convolution must be understood after prolonging f as an even function on $\mathbb{R}$. The covariance function Γ_ε of $X^\varepsilon(t)$ satisfies $\Gamma_\varepsilon(t) = \Gamma(t)\psi(\varepsilon t)$. This implies that the process $X^\varepsilon(t)$ is $\frac{1}{\varepsilon}$-dependent. We have the following proposition:

Proposition 8. *Let $\mathcal{X}$ be a centered stationary Gaussian process with spectral density $f(\lambda)$ and covariance function Γ with $\Gamma^\ell \in L^1(\mathbb{R})$, ℓ positive integer. Let $X_\varepsilon(t)$ be defined by (10). Then*

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \mathrm{E}\left[\frac{1}{\sqrt{t}}\int_0^t \left(H_\ell(X(s)) - H_\ell(X^\varepsilon(s))\right)ds\right]^2 = 0. \tag{11}$$

Theorem 9. *Let $\mathcal{X}$ be a Gaussian process satisfying the hypotheses of Proposition 8 and F a function in $L^2(\phi(x)dx)$ with Hermite rank $\ell \geq 1$. Then, as $t \to +\infty$,*

$$\sqrt{t}T_t = \frac{1}{\sqrt{t}}\int_0^t F(X(s))ds \to N(0, \sigma^2(F)) \text{ in distribution}$$

where

$$\sigma^2(F) := 2\sum_{k=\ell}^\infty a_k^2 k! \int_0^\infty \Gamma^k(s)ds.$$

Proof:

Define $F_M := \sum_{n=\ell}^M a_n H_n(x)$ and $T_t^M := \frac{1}{t}\int_0^t F_M(X(s))ds$. Let $M = M(\delta) > \ell$ such that

$$2\sum_{k=M+1}^\infty a_k^2 < \delta.$$

Using Mehler's formula, we get

$$t\ \mathrm{Var}(T_t - T_t^M) = 2\sum_{k=M}^{\infty} c_k^2 k! \int_0^t (1-\frac{s}{t})\Gamma^k(s)ds \le 2\sum_{k=M}^{\infty} c_k^2 k! \int_0^{\infty} |\Gamma|^k(s)ds$$
$$< \delta \int_0^{\infty} |\Gamma|^{\ell}(s)ds.$$

Since δ is arbitrary, we only need to prove the asymptotic normality for T_t^M. Let us introduce

$$T_t^{M,\varepsilon} = \frac{1}{t}\int_0^t F_M(X^{\varepsilon}(s))ds,$$

where $X_{\varepsilon}(t)$ has been defined in (10). By Proposition 8 recalling that for $k \ge l$, Γ^k is in $L^1(\mathbb{R})$ since Γ^{ℓ} is, we obtain:

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} t\ \mathrm{Var}(T_t^M - T_t^{M,\varepsilon}) = 0.$$

Now Theorem 7 for m- dependent sequences implies that $\sqrt{t}\, T_t^{M,\varepsilon}$ is asymptotically normal. Notice that

$$\sigma_{M,\varepsilon} := \lim_{t\to\infty} t\,\mathrm{Var}(T_t^{M,\varepsilon}) = 2\sum_{k=0}^{M} a_k^2 k! \int_0^{\frac{1}{\varepsilon}} \Gamma_{\varepsilon}^k(s)ds$$

and that $\sigma_{M,\varepsilon} \to \sigma^2(F)$ when $\varepsilon \to 0$ and $M \to \infty$, giving the result. ∎

3.2. **Hermite expansion for crossings of regular processes.** Our aim is to extend the result above to crossings. Let $X(t)$ be a centered stationary Gaussian process. With no loss of generality for our purposes, we assume that $\Gamma(0) = -\Gamma''(0) = 1$ and $\Gamma(t) \ne \pm 1$ for $t \ne 0$. We also assume Geman's Condition (6).

$$\Gamma(t) = 1 - t^2/2 + \theta(t) \quad \text{with} \int \frac{\theta'(t)}{t^2}dt \text{ converges at } 0^+.$$

We define the following expansions

$$x^+ = \sum_{k=0}^{\infty} a_k H_k(x), \quad x^- = \sum_{k=0}^{\infty} b_k H_k(x), \quad |x| = \sum_{k=0}^{\infty} c_k H_k(x). \tag{12}$$

We have $a_1 = 1/2,\ b_1 = -1/2,\ c_1 = 0$ and using integration by parts for $k > 2$:

$$a_k = \frac{1}{k!}\int_0^{+\infty} xH_k(x)\varphi(x)dx = \frac{1}{k!\sqrt{2\pi}}H_{k-2}(0).$$

The classical properties of Hermite polynomials easily imply that for positive k:

$$a_{2k+1} = b_{2k+1} = c_{2k+1} = 0,$$
$$a_{2k} = b_{2k} = \frac{(-1)^{k+1}}{\sqrt{2\pi}2^k k!(2k-1)},$$
$$c_{2k} = 2a_{2k}.$$

We have the following Hermite expansion for the number of up-crossings:

Theorem 10. *Under the conditions above,*

$$U_u := U_u(X,[0,T]) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_j(u) a_k \int_0^T H_j(X(s))H_k(X'(s))ds \ \text{a.s.}$$

where $d_j(u) = \frac{1}{j!}\phi(u)H_j(u)$ *and* a_k *is defined by (12). We have similar results, replacing* a_k *by* b_k *or* c_k*, for the number* $D_u([0,T])$ *of down-crossings and for the total number of crossings* $N_u([0,T])$*.*

Proof : Let $g(.) \in L^2(\phi(x)dx)$ and define the functional

$$T_g^+(t) = \int_0^t g(X(s))X'^+(s)ds.$$

The convergence of the Hermite expansion implies that a.s.

$$T_g^+(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} g_j\, a_k \int_0^t H_j(X(s))H_k(X'(s))ds, \tag{13}$$

where the g_j's are the coefficients of the Hermite expansion of g. Using that for each s, $X(s)$ and $X'(s)$ are independent, we get:

$$\mathrm{E}\Big[\int_0^t \Big[g(X(s))(X'(s))^+ - \sum_{j,k\geq 0: k+j\leq Q} g_j a_k H_j(X(s))H_k(X'(s))\Big]ds\Big]^2 \leq (const)t^2 \sum_{j,k\geq 0: k+j\geq Q} j! g_j^2 k! a_k^2. \tag{14}$$

On the other hand, using the Geman condition

$$\nu_2(u,T) := \mathrm{E}\big(U_u([0,T])(U_u([0,T])-1)\big) < +\infty.$$

For every T, $\nu_2(u,T)$ is a bounded continuous function of u and the same holds true for $\mathrm{E}(U_u^2)$. Let us now define

$$U_u^\delta := \frac{1}{2\delta}\int_0^T \mathbb{1}_{|X(t)-u|\leq\delta} X'^+(t)dt.$$

In our case, hypotheses of Proposition 1 are a.s. satisfied. The result can be easily extended to up-crossings, showing that

$$U_u^\delta \to U_u \text{ a.s. as } \delta \to 0.$$

By Fatou's Lemma

$$\mathrm{E}\big((U_u)^2\big) \leq \liminf_{\delta\to 0} \mathrm{E}\big((U_u^\delta)^2\big).$$

To obtain an inequality in the opposite sense, we use the Banach formula (Proposition 2). To do that, notice that this formula remains valid if one replaces in the left-hand side the total number of crossings by the up-crossings and in the right-hand side $|f'(t)|$ by $f'^+(t)$. So, on applying it to the random path $X(.)$, we see that:

$$U_u^\delta = \frac{1}{2\delta}\int_{u-\delta}^{u+\delta} U_x dx.$$

Using Jensen's inequality,

$$\limsup_{\delta\to 0} \mathrm{E}\big((U_u^\delta)^2\big) \leq \limsup_{\delta\to 0} \frac{1}{2\delta}\int_{u-\delta}^{u+\delta} \mathrm{E}\big((U_x)^2\big)dx = \mathrm{E}\big((U_u)^2\big)$$

So, $\mathrm{E}\big((U_u^\delta)^2\big) \to \mathrm{E}\big((U_u)^2\big)$ and since the random variables involved are non-negative, a standard argument of passage to the limit based upon Fatou's Lemma shows that $U_u^\delta \to U_u$ in L^2.

We now apply (13) to U_u^δ.

$$U_u^\delta = \sum_{j,k=0}^{\infty} d_j^\delta(u) a_k \zeta_{jk}, \tag{15}$$

where $d_j^\delta(u)$ are the Hermite coefficients of the function $x \rightsquigarrow \frac{1}{\delta}\mathbb{1}_{\|x-u\|\leq\delta}$ and

$$\zeta_{jk} = \int_0^T H_j(X(s))H_k(X'(s))ds.$$

Notice that

$$d_j^\delta(u) \to \frac{1}{j!}\phi(u)H_j(u) = d_j(u). \tag{16}$$

This implies that:

$$U_u = \sum_{q=0}^{\infty} \sum_{j+k=q} d_j(u) a_k \zeta_{jk}. \tag{17}$$

∎

Theorem 11. *Let $\{X(t) : t \in \mathbb{R}\}$ be a centered stationary Gaussian process verifying the conditions at the beginning of this subsection. Furthermore, let us assume that:*

$$\int_0^{+\infty} |\Gamma(t)|dt, \int_0^{+\infty} |\Gamma'(t)|dt, \int_0^{+\infty} |\Gamma''(t)|dt < \infty. \tag{18}$$

Let $\{g_k\}_{k=0,1,2,\ldots}$ a sequence of coefficients which satisfies $\sum_0^{+\infty} g_k^2 k! < \infty$. Put:

$$F_t := \frac{1}{\sqrt{t}} \sum_{k,j\geq 0} g_j a_k \int_0^t H_j(X(s))H_k(X'(s))ds$$

where a_k has been defined in (12). Then

$$F_t - \mathrm{E}(F_t) \to N(0,\sigma^2) \text{ in distribution as } t \to +\infty$$

where

$$0 < \sigma^2 = \sum_{q=1}^{\infty} \sigma^2(q) < \infty,$$

and

$$\sigma^2(q) := 2\sum_{k=0}^{q}\sum_{k'=0}^{q} a_k a_{k'} g_{q-k} g_{q-k'} \times \int_0^{+\infty} \mathrm{E}\big[H_{q-k}(X(0))H_k(X'(0))H_{q-k'}(X(s))H_{k'}(X'(s))\big]ds.$$

The integrand in the right-hand side of this formula can be computed using Lemma 5. Similar results exist, mutatis mutandis, for the sequences $\{b_k\}$ and $\{c_k\}$.

A consequence is

Corollary 12. *If the process $X(t)$ satisfies the conditions of Theorem 11 then, as $T \to +\infty$*

$$\frac{1}{\sqrt{T}}\Big(U_u\big([0,T]\big) - T\frac{e^{-u^2/2}}{2\pi}\Big) \to N(0,\sigma_1^2) \textit{ in distribution}$$
$$\frac{1}{\sqrt{T}}\Big(N_u\big([0,T]\big) - T\frac{e^{-u^2/2}}{\pi}\Big) \to N(0,\sigma_2^2) \textit{ in distribution,}$$

where σ_1^2 and σ_2^2 are finite and positive.

Remark The result of Theorem 11 is in fact true under weaker hypotheses namely

$$\int_0^{+\infty} |\Gamma(t)|dt < \infty\,, \qquad \int_0^{+\infty} |\Gamma''^2(t)|dt < \infty\,,$$

see Theorem 1 of Kratz and León [11] or Kratz [9].See also Azaïs and Leon [1] for another generalization where the integral $\int_{\mathbb{R}} \Gamma(t)dt$ is defined only in a generalized sense. Our stronger hypotheses make it possible to make a shorter proof.

Proof of the theorem:

Since Γ is integrable, the process $\mathcal{X}$ admits a spectral density. The hypotheses and the Riemann-Lebesgue lemma imply that:

$$\Gamma^{(i)}(t) \to 0 \quad i = 0,1,2 \quad \text{as } t \to +\infty.$$

Hence, we can choose T_0 so that for $t \geq T_0$

$$\overline{\Gamma}(t) := \sup\{|\Gamma(t)|, |\Gamma'(t)|, |\Gamma''(t)|\} \leq 1/4. \tag{19}$$

Step 1. In this step we prove that one can choose Q large enough (and that doesn't depend on t) so that F_t can be replaced with an arbitrarily small error (in the L^2 sense) by its components in the first Q chaos

$$F_t^Q := \frac{1}{\sqrt{t}}\sum_{q=0}^{Q} G_t^q \quad \text{with } G_t^q := \sum_{k=0}^{q} g_{q-k}a_k \int_0^t H_{q-k}(X(s))H_k(X'(s))ds.$$

Let us consider

$$\frac{1}{t}\mathrm{E}\big((G_t^q)^2\big) = 1/t \sum_{k,k'=0}^{q} g_{q-k}a_k g_{q-k'}a_{k'} \int_0^t dt_1 \cdot \int_0^t \mathrm{E}\big(H_{q-k}(X(t_1))H_k(X'(t_1))H_{q-k'}(X(t_2))H_{k'}(X'(t_2))dt_2. \tag{20}$$

To give an upper-bound for this quantity we split it into two parts.

The part corresponding to $|t_1 - t_2| \geq T_0$ is bounded, using Lemma 5, by

$$
\begin{aligned}
(21) \quad & (const) \sum_{k,k'=0}^{q} |g_{q-k}||a_k||g_{q-k'}||a_{k'}| \\
& \times \int_{T_0}^{t} \sum_{(d_1,d_2,d_3,d_4)\in Z} \frac{k!(q-k)!k'!(q-k')!}{d_1!d_2!d_3!d_4!} |\Gamma(s)|^{d_1} |\Gamma'(s)|^{d_2+d_3} |\Gamma''(s)|^{d_4} ds \\
& \leq (const) \sum_{k,k'=0}^{q} |g_{q-k}||a_k||g_{q-k'}||a_{k'}| \\
& \times \int_{T_0}^{t} \sum_{(d_1,d_2,d_3,d_4)\in Z} \frac{k!(q-k)!k'!(q-k')!}{d_1!d_2!d_3!d_4!} (\frac{1}{4})^{(q-1)} \overline{\Gamma}(s) ds,
\end{aligned}
$$

where Z is as in Lemma 5, setting $r_1 = q-k, r_2 = k, r_3 = q-k', r_4 = k'$.

Remarking that $\sup_d \frac{1}{d!(k-d)!} \leq \frac{2^k}{k!}$ it follows that $\frac{k!(q-k)!k'!(q-k')!}{d_1!d_2!d_3!d_4!}$ in (21) is bounded above by $2^q(k')!(q-k')!$ or $2^q(k)!(q-k)!$ depending on the way we group terms. As a consequence it is also bounded above by $2^q\sqrt{(k')!(q-k')!(k)!(q-k)!}$ and the right-hand side of (21) is bounded above by

(22)

$$
\begin{aligned}
& (const) \sum_{k,k'=0}^{q} |g_{q-k}||a_k||g_{q-k'}||a_{k'}| q 2^{-q} \sqrt{(k')!(q-k')!(k)!(q-k)!} \int_0^{+\infty} \overline{\Gamma}(t) dt \\
& \qquad \leq (const) \sum_{k,k'=0}^{q} |g_{q-k}||a_k||g_{q-k'}||a_{k'}| \sqrt{(k')!(q-k')!(k)!(q-k)!}
\end{aligned}
$$

where we have used that the number of terms in Z is bounded by q.

On the other hand, the integration region in (20) corresponding to $|t_1 - t_2| \leq T_0$ can be covered by at most $[t/T_0]$ squares of size $2T_0$. Using Jensen's inequality as we did for the proof of (14) we obtain:

$$
\mathrm{E}\Big(\big(G^q_{2T_0}\big)^2 \Big) \leq (const) T_0^2 \sum_{k=0}^{q} (q-k)!k! g_{q-k}^2 a_k^2. \tag{23}
$$

Finally,

$$
\frac{1}{t}\mathrm{E}\Big(\big(G^q_{t}\big)^2 \Big) \leq (const) \sum_{k=0}^{q} (q-k)!k! g_{q-k}^2 a_k^2,
$$

which is the general term of a convergent series. This proves also that σ^2 is finite.

Step 2. Let us prove that $\sigma^2 > 0$. It is sufficient to prove that $\sigma^2(2) > 0$. Recall that $a_1 = 0$ so that

$$\begin{aligned}(24)\quad \sigma^2(2) = a_0^2 g_2^2 \int_0^{+\infty} &\mathrm{E}\big(H_2(X(0))H_2(X(s))ds \\ &+ a_2^2 g_0^2 \int_0^{+\infty} \mathrm{E}\big(H_2(X'(0))H_2(X'(s))ds \\ &\quad + 2a_0 g_2 a_2 g_0 \int_0^{+\infty} \mathrm{E}\big(H_2(X(0))H_2(X'(s))ds.\end{aligned}$$

Using the Mehler formula

$$\begin{aligned}(25)\qquad \sigma^2(2) &= 2a_0^2 g_2^2 \int_0^{+\infty} \Gamma^2(s)ds \\ &\quad + 2a_2^2 g_0^2 \int_0^{+\infty} (\Gamma''(s))^2 ds + 4a_0 g_2 a_2 g_0 \int_0^{+\infty} (\Gamma'(s))^2 ds \\ &= \int_{-\infty}^{+\infty} \big(\lambda^4 a_0^2 g_2^2 + \lambda^2 2a_0 g_2 a_2 g_0 + a_0^2 g^2\big) f^2(\lambda) d\lambda \\ &= \int_{-\infty}^{+\infty} \big(\lambda^2 a_2 g_0 + a_0 g_2\big)^2 f^2(\lambda) d\lambda > 0.\end{aligned}$$

Step 3. We define $\psi(.) = K\big(\mathbb{1}_{[1/4,1/4]}\big)^{*4}$, where the constant K is chosen such that $\psi(0) = 1$. Then we define $X^\varepsilon(t)$ using (10). The new definition of $\psi(.)$ ensures now that $X^\varepsilon(t)$ is differentiable. Define

$$F_t^{Q,\varepsilon} := \frac{1}{\sqrt{t}} \sum_{q=0}^{Q} G_t^{q,\varepsilon},$$

with

$$G_t^{q,\varepsilon} = \sum_{k=0}^{q} g_{q-k} a_k \int_0^t H_{q-k}(X^\varepsilon(s)) H_k\big((X^\varepsilon)'(s)\big) ds.$$

In this step, we prove that F_t^Q can be replaced, with an arbitrarily small error if ε is small enough, by $F_t^{Q,\varepsilon}$. Since the expression of F_t^Q involves only a finite number of terms having the form:

$$K^0_{q-k,k} := \frac{1}{\sqrt{t}} \int_0^t H_{q-k}(X(s)) H_k\big(X'(s)\big) ds$$

if ε is small enough, one can replace with an arbitrarily small error by

$$K^\varepsilon_{q-k,k} := \frac{1}{\sqrt{t}} \int_0^t H_{q-k}(X^\varepsilon(s)) H_k\big((X^\varepsilon)'(s)\big) ds.$$

For that purpose we study

$$\begin{aligned}
&\mathrm{E}(K^0_{q-k,k} - K^{\varepsilon}_{q-k,k})^2 \\
&= 2\int_0^t \frac{t-s}{t} \mathrm{E}\Big[H_{q-k}(X(0))H_k\big(X'(0)\big)H_{q-k}(X(s))H_k\big(X'(s)\big)\Big] \\
&\quad + \mathrm{E}\Big[H_{q-k}(X^{\varepsilon}(0))H_k\big((X^{\varepsilon})'(0)\big)H_{q-k}(X^{\varepsilon}(s))H_k\big((X^{\varepsilon})'(s)\big)\Big] \\
&\quad - 2\mathrm{E}\Big[H_{q-k}(X(0))H_k\big(X'(0)\big)H_{q-k}(X^{\varepsilon}(s))H_k\big((X^{\varepsilon})'(s)\big)\Big]ds.
\end{aligned}$$

Consider the computation of terms of the kind

$$\int_0^t \frac{t-s}{t} \mathrm{E}\Big[H_{q-k}(Y_1(0))H_k\big(Y_1'(0)\big)H_{q-k}(Y_2(s))H_k\big(Y_2'(s)\big)\Big]ds \tag{26}$$

where the processes $Y_1(t)$ and $Y_2(t)$ are chosen among $\{X(t), X^{\varepsilon}(t)\}$. It suffices to prove that all these terms have the same limit, as $t \to +\infty$ and then $\varepsilon \to 0$ whatever the choice is.

Applying Lemma 5, the expectation in(26) is equal to

$$\int_0^t \frac{t-s}{t} \sum_{d_1,\dots,d_4 \in Z} \frac{(q-k)!^2 k!^2}{d_1!d_2!d_3!d_4!} (\rho(s))^{d_1}(\rho'(s))^{d_2}(-\rho'(s))^{d_3}(-\rho''(s))^{d_4} ds,$$

where $\rho(.)$ is the covariance function between the processes Y_1 and Y_2 and Z is defined as in Lemma 5. Again, since the number of terms in Z is finite, it suffices to prove that

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \int_0^t \frac{t-s}{t} (\rho(s))^{d_1}(\rho'(s))^{d_2+d_3}(\rho''(s))^{d_4} ds,$$

where $(d_1,\dots,d_4)$ is chosen in Z, does not depend on the way to choose Y_1 and Y_2. ρ is the Fourier transform of (say) $g(\lambda)$ which is taken among $f(\lambda)$; $f * \widehat{\psi}_{\varepsilon}(\lambda)$ or $\sqrt{f(\lambda)}\sqrt{f * \widehat{\psi}_{\varepsilon}(\lambda)}$. Define $\overline{g}(\lambda) = i\lambda g(\lambda)$ and $\overline{\overline{g}}(\lambda) = -\lambda^2 g(\lambda)$. Then $(\rho(s))^{d_1}(\rho'(s))^{d_2+d_3}(\rho''(s))^{d_4}$ is the Fourier transform of the function

$$h(\lambda) = g^{*d_1}(\lambda) * \overline{g}^{*(d_2+d_3)}(\lambda) * \overline{\overline{g}}^{*d_4}(\lambda).$$

The continuity and boundedness of f imply that all the functions above are bounded and continuous. The Fubini theorem shows that

$$\int_0^t \frac{t-s}{t} \rho(s)^{d_1}\rho'(s)^{d_2+d_3}(\rho''(s))^{d_4} ds = \int_{-\infty}^{+\infty} \frac{1-\cos\lambda}{\lambda^2} h(\frac{\lambda}{t}),$$

As $t \to +\infty$, the right-hand side converges, using dominated convergence, to

$$\int_{-\infty}^{+\infty} \frac{1-\cos\lambda}{\lambda^2} h(0) d\lambda.$$

The continuity of f now gives the result, as in Proposition 8.

∎

Proof of Corollary 12:

Some attention must be payed to the fact that the coefficients

$$d_j(u) = \frac{1}{j!}\phi(u)H_j(u)$$

do not satisfy $\sum_{j=0}^{\infty} j!d_j^2(u) < \infty$. They only satisfy the relation

$$j!d_j^2(u) \text{ is bounded} \tag{27}$$

First, considering the bound given by the right-hand side of (22), we can improve it by reintroducing the factor $q2^{-q}$ that had been bound by 1. We get that in its new expression this right-hand side is bounded by

$$(const)q2^{-q} \sum_{k,k'=0}^{q} |d_{q-k}(u)||a_k||d_{q-k'}(u')||a_{k'}|\sqrt{(k')!(q-k')!(k)!(q-k)!}$$
$$\leq (const)q^2 2^{-q} \sum_{k=0}^{q} \big(d_{q-k}(u)\big)^2 a_k^2 (k)!(q-k)!$$
$$\leq (const)q^2 2^{-q} \sum_{k=0}^{q} a_k^2 k! \leq (const)q^2 2^{-q}.$$

Second we have to replace the bound (23). Since the series in (17) is convergent $\mathrm{E}\Big(\big(G_{2T_0}^q\big)^2\Big)$ is the term of a convergent series and this in enough to conclude. ∎

.

4. Applications and extensions

In an unpublished manuscript, Stephane Mourareau has extended the result of Corollary 12 to the case of moving level u_T.

Theorem 13. *Let u_T be a moving level that tends to infinity with T. Suppose that*

- *The process $X(t)$ is m-dependent*
- $$\mathrm{E}(U_t) \to \infty$$

Then

$$\frac{1}{\sqrt{T\phi(u_T)}}\Big(U_{u_T}(T) - \sqrt{\frac{\lambda_2}{2\pi}}T\phi(u_T)\Big) \Rightarrow \mathcal{N}\big(0, \frac{\lambda_2}{2\pi}\big)$$

The variance is now simple and explicit and it corresponds to the Poissonian limit (the variance is equal to the expectation) known as the Vlokonskii- Rozanov theorem.

Theorem 14. *Assume the conditions of Theorem 11 except* (18) *which is now replaced by the very weak Berman's condition*

$$\Gamma(\tau)\log(\tau) \to 0 \text{ as } \tau \to \infty.$$

Let u_T be a movinf level such that $\mathrm{E}\big(U_{u_t}\big) = \lambda$ where λ is some constant. Then U_{u_t} converges to a Poisson distribution with parameter λ.

This is a simplified version, the full one establishes a functional convergence of the point process itself.

4.1. **Random trigonometric polynomials.** Let $X(t)$ be the stochastic process with covariance

$$\Gamma(t) = \frac{\sin(t)}{t}$$

Since the covariance is not summable in the Lebesgue sense, it does not satisfy strictly the conditions of Corollary 12. But in fact the integral

$$\int_{\mathbb{R}} \Gamma(t)dt$$

can be defined by passage to the limit and it can be checked that the result holds true.

Let $X_N(t)$ the sequences of random trigonometric polynomials given by

$$X_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (a_n \sin nt + b_n \cos nt),$$

where the a_n, b_n's are independent standard normal.

it is easy to check that for each N, $X_N(t)$ is a stationary Gaussian process with covariance:

$$\Gamma_{X_N}(\tau) := \mathbb{E}[X_N(0)X_N(\tau)] = \frac{1}{N}\sum_{n=1}^{N} \cos n\tau = \frac{1}{N}\cos(\frac{(N+1)\tau}{2})\frac{\sin(\frac{N\tau}{2})}{\sin\frac{\tau}{2}}. \tag{28}$$

We define the process

$$Y_N(t) = X_N(t/N),$$

with covariance

$$\Gamma_{Y_N}(\tau) = \Gamma_{X_N}(\tau/N).$$

The convergence of the Rieman sum to the intergral implies that

$$\Gamma_{Y_N}(\tau) \to \Gamma(\tau) := \sin(\tau)/\tau \text{ as } N \to +\infty$$

And the have the same type of control for the derivatives. The main argument of Azaïs and León [1] is a construction of the process $X_N(t)$ as well as the limit $X(t)$ in the same probability space to get that the Central limit theorem for the crossings of $X(t)$ pass to those of $X_N(t)$. It gives a generalization of a paper by Grandville and Wigman [8]

Theorem 15. *With the notation above*

(1) $$\frac{1}{\sqrt{N\pi}}\big(N^{Y_N}_{[0,N\pi]}(u) - \mathrm{E}(N^{Y_N}_{[0,N\pi]}(u))\big) \Rightarrow N(0, \frac{1}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty}\sigma_q^2(u)),$$

(2) $$\frac{1}{\sqrt{2N\pi}}\big(N^{Y_N}_{[0,2N\pi]}(u) - \mathrm{E}(N^{Y_N}_{[0,2N\pi]}(u))\big) \Rightarrow N(0, \frac{2}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty}\sigma_q^2(u)),$$

where $\Rightarrow$ is the convergence in distribution as $N \to \infty$ and $\sigma_q^2(u)$ is the variance of the part in the qth chaos.

4.2. **Specular points.** A different case of central limit theorem is given by the number of specular points. These are point of the surface of the sea that appear in bright on a photo. We use a cylinder model: time is fixed; the variation of the elevation of the sea $W(x)$ as a function of the space variable x is modeled by a smooth stationary Gaussian process; as a function of the second space variable y the elevation of the sea is supposed to be constant.

Suppose that a source of light is located at $(0, h_1)$ and that an observer is located at $(0, h_2)$ where h_1 and h_2 are big with respect to $W(x)$ and x. Only the variable x has to be taken into account and the following approximation, was introduced long ago by Longuett-Higgins [14]: the point x is a specular point if

$$W'(x) \simeq kx, \text{ with } k := \frac{1}{2}\Big(\frac{1}{h_1} + \frac{1}{h_2}\Big).$$

This is a non stationary case: there are more specular points underneath the observer. In particular if $SP(I)$ s the number of specular points contained in the interval I,

$$\mathrm{E}(SP(I)) = \int_I G(-k, \sqrt{\lambda_4})\frac{1}{\sqrt{\lambda_2}}\varphi(\frac{kx}{\sqrt{\lambda_2}})dx, \tag{29}$$

where λ_2, λ_4 are the spectral moments of order 2 and 4 respectively that are assumed to be finite; $G(\mu, \sigma) := \mathrm{E}(|Z|)$, Z with distribution $N(\mu, \sigma^2)$.

An easy consequence of that formula is that

$$\mathrm{E}(SP) := \mathrm{E}(SP(\mathbb{R})) = \frac{G(k, \sqrt{\lambda 4})}{k} \simeq \sqrt{\frac{2\lambda_4}{\pi}}\frac{1}{k},$$

as k tends to 0.

As a consequence the number of specular point is almost surely finite and the Central Limit Theorem may only happen in the case where $k \to 0$,i.e. when the locations of the observer an the source of light are infinitely far from the surface of the sea.

The central limit theorem is now established using Lyapounov type conditions for Lindeberg type Central Limit Theorem for triangular arrays.

Theorem 16. *Under some conditions (see Azaïs León and Wschebor* [2] *for details), as* $k \to 0$,

$$\frac{S - \sqrt{\frac{2\lambda_4}{\pi}}\frac{1}{k}}{\sqrt{\theta/k}} \Rightarrow N(0,1), \textit{ in distribution,}$$

where θ *is some (complicated) constant.*

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