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ENUMERATIVE GEOMETRY IN SPACES OF FOLIATIONS

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Introduction

The aim of this text is to present some applications of intersection theory to the global study of holomorphic foliations in projective spaces.

Holomorphic foliations are an offspring of the geometric theory of polynomial differential equations. Following the trend of many branches in Mathematics, interest has migrated to global aspects. Instead of focusing on just one curve, or surface, or metric, or differential equation, try and study their family in a suitable parameter space. The geometry within the parameter space of the family acquires relevance. For instance, the family of hypersurfaces of a given degree correspond to points in a suitable projective space; geometric conditions imposed on hypersurfaces, *e.g.,* requiring it to be singular, usually correspond to interesting subvarieties in the parameter space, *e.g.,* the discriminant. Hilbert schemes have their counterpart in the theory of polynomial differential equations, to wit, the spaces of foliations.

The last few years have witnessed an important development of the study of holomorphic foliations. Works of Jouanolou, [32],[33] Cerveau, [5],[6] Lins-Neto, [37],[38], Pereira [7], [9],[41], Cukierman [10],[11],[12], Calvo-Andrade [2],[3] Gómez-Mont [26], [23],[24],[25] and others have focused on global aspects, clarifying questions regarding the (non–)existence of algebraic leaves, description of components for the spaces of foliations of codimension ≥ 1, etc.

Our point of view follows the line of classical enumerative geometry, which treats questions such as: *How many plane curves pass through an appropriate number of points in general position? How many space curves varying in a given family are incident to an appropriate number of lines in general position? Find the degree of the space of planes curves having a singularity of given order; compute the degree of the variety of hypersurfaces containing linear subspaces, or conics, or twisted cubics, and so on ...*

In this work we consider similar questions for holomorphic foliations.

Holomorphic foliations of degree d on the complex projective plane $\mathbb{P}^2$ are defined by nonzero twisted 1–forms, $\omega = \sum a_i dZ_i$, with homogeneous polynomials $a_i(Z_0, Z_1, Z_2)$ of degree $d+1$, up to scalar multiples, satisfying $\sum a_i Z_i = 0$. The parameter space of foliations of degree d is a projective space $\mathbb{P}^N$ (the coordinates being the coefficients of the a_is) (cf. (2.4), (2.6), p. 48). The scheme of singularities of ω is defined by the homogeneous ideal generated by the a_i.

It is well known that a *general* foliation of degree d on $\mathbb{P}^2$ has exactly d^2+d+1 singularities, all non-degenerate. So it makes sense to try and study the geometry of the set of foliations with degenerate singularities.

We show how to find the degrees of the subvarieties of $\mathbb{P}^N$ corresponding to foliations displaying certain degenerate singularities. Given an integer $k \geq 2$ we study the locus, $\mathbb{M}_k \subset \mathbb{P}^N$, of foliations with a singularity of order $\geq k$. These are foliations defined in local coordinates by a holomorphic 1–form that can be written as $\omega = a_k dx + b_k dy +$higher order terms, with $a_k(x,y), b_k(x,y)$ homogeneous polynomials of degree k. It turns out that $\mathbb{M}_k$ is the birational image of an explicit projective bundle over $\mathbb{P}^2$. Using tools from intersection theory, we find a formula for the degree of $\mathbb{M}_k$.

Another interesting type of non-generic foliation presents a so called *dicritical* singularity of order k: require $a_k x + b_k y$ to vanish. This defines a closed subset $\mathbb{D}_k \subset \mathbb{M}_k$.

A geometric interpretation for the degree of the above subvarieties is as follow: Requiring a leaf of a foliation to be tangent to a fixed line at a given point defines a hyperplane in the parameter space $\mathbb{P}^N$. Therefore, the degree of each of the loci

$\mathbb{D}_k \subset \mathbb{M}_k \subset \mathbb{P}^N$ can be reinterpreted loosely as the number of foliations with a singularity of the specified type and further tangent to the appropriate number of flags (point, line) in $\mathbb{P}^2$. It turns out that the degrees of $\mathbb{D}_k, \mathbb{M}_k$ are expressed as explicit polynomials in k, d.

This fits nicely into the tradition of classical enumerative geometry: answers to questions such as determining the number of plane algebraic curves that have singularities of prescribed orders, besides having to pass through an appropriate number of points in general position, are often given by so called "node" polynomials. There is a wealth of results and conjectures on generating functions for counting suitably singular members of linear systems of curves on surfaces, cf. Götsche [22], Kleiman and Piene [35]. We hope similar results can be formulated in the setting of foliations.

Continuing the analogy with enumerative geometry, let us recall that, while a *general* surface of degree $d \geq 4$ in $\mathbb{P}^3$ contains no line –in fact, only complete intersection curves are allowed, those that *do contain* some line correspond to a subvariety of codimension $d-3$ and degree $\binom{d+1}{4}(3d^4+6d^3+17d^2+22d+24)/4!$ in a suitable $\mathbb{P}^N$.

Similarly, motivated by Jouanolou's celebrated theorem to the effect that a general holomorphic foliation, say in $\mathbb{P}^2$, of degree $d \geq 2$ has no algebraic leaf [32], we show that those foliations that *do have*, *e.g.*, an invariant line, correspond to a subvariety of codimension $d-1$ and degree $3\binom{d+3}{4}$ in a suitable $\mathbb{P}^N$.

The rest of these notes is dedicated to the study of the spaces of foliations having certain invariant algebraic subvarieties. First we impose linear subspaces, and then quadrics.

The general philosophy is to find a suitable complete parameter space for the families of foliations satisfying some of the above conditions: bad singularity, or invariant subvariety of a given type. Then, using techniques of intersection theory we compute their dimension and degree. In the case of foliations with a degenerate singularity (resp. with some invariant linear subspace) the construction of the parameter space is very explicit. In fact, in the case of singularities, we describe the parameter space as the image in the space of foliations of natural projective bundles over $\mathbb{P}^2$. For invariant linear subspaces, we also get a projective bundle over a Grassmannian. Actually, these bundles are projectivizations of vector bundles, the characteristic classes of which we are able to determine.

The case of foliations having an invariant quadric turns out to be subtler and hints at the difficulties to handle degrees higher than one. We make and do invoking the classical complete quadrics. In short, we blowup the projective space parametrizing the family of quadrics along the locus corresponding to singular quadrics. In this way we find a compactification of the space of foliations having an invariant quadric, with enough information to compute its degree.

Next we survey the contents of each section.

In Section 1 we give, for the reader's convenience, a brief introduction to intersection theory in projective spaces and Grassmannians. We define the Chow group of a scheme and describe it for the case of projective bundles and Grassmannians. We also review Fulton's construction of Chern and Segre classes asociated to a vector bundle. We list some of their properties that should help the reader to understand the computations developed in the rest of this text.

Section 2 contains the basic notions of holomorphic foliations of dimension one, as well as of codimension one, in projective spaces. We give references to the literature and definitions and results that will be needed in the sequel.

In Section 3 we find formulas for the dimension and degree of the space of foliations of degree d in $\mathbb{P}^2$ that have a degenerate singularity. In the first part we find parameter spaces for foliations having a singularity of given order $k \geq 2$. In the second part we study foliations with a dicritical singularity of order k.

Section 4 is dedicated to the study of foliations having some invariant algebraic subvariety. In the first part we find parameter spaces for the variety of foliations in $\mathbb{P}^n$ having an invariant linear subspace of given dimension. Using this description we obtain formulas for its dimension and degree. In the second part we find a compactification of the space of foliations having a smooth invariant conic in $\mathbb{P}^2$. In this way, we get formulas for its dimension and degree.

We include an appendix intended to be a glossary of basic concept and constructions needed in the text, such as vector bundles, Cartier divisors, Grassmannians, etc...

There are many natural generalizations of the material covered here. For instance, we could impose flags of invariant subvarieties. Indeed, the general foliation in $\mathbb{P}^3$ that leaves invariant some plane does not need to leave any invariant curve therein. So it makes sense to ask for the degree of the space of foliations that leave invariant, say a flag plane $\supset$ line, or plane $\supset$ conic, ...

Another interesting direction is to study the imposition, say of a given class of curves, to be contained in the scheme of singularities, cf. G.N. Costa, [8].

Extension to the case of higher order differential equations can be pursued following the ideas in M. Falla's thesis, [15]. In Chapter 5 of [20], we find formulas for the degree and codimension of the space of second order equations having a line as solution.

Most of the matterial covered here is taken from my thesis [16], and was published in [20], with my advisor Israel Vainsencher, cf. also [17], [18].

1. Intersection Theory in Grassmannians

In this Section we introduce briefly the Chow group of a scheme. We define Chern and Segre classes associated to a vector bundle, and discuss some of their properties. We give an explicit description of the Chow groups of the projective space $\mathbb{P}^n$ and the Grassmannian $\mathbb{G}(k,n)$, that is all we will need in the course. The reference for this material is [21], [46].

All schemes are of finite type over a field, usually the complex numbers $\mathbb{C}$. Variety means reduced and irreducible (*i.e.,* integral) scheme; likewise for subvarieties.

1.1. Cycles.

1.1.1. Definition. *Let X be a scheme. The* **group of cycles of dimension** k *of X is the free abelian group generated by the closed subvarieties of dimension k in X. It will be denoted by $\mathcal{C}_kX$. The* **group of cycles** *is the graded group*

$$\mathcal{C}_* = \bigoplus_k \mathcal{C}_kX.$$

By definition, each $k-$cycle $c \in \mathcal{C}_k X$ can be written in a unique way as a linear combination with coefficients in $\mathbb{Z}$,

$$c = \sum_V n_V \cdot V,$$

where V runs in the collection of closed subvarieties of X of dimension k. Here, the coefficient $n_V \in \mathbb{Z}$ is zero except for finitely many V's.

Recall that if $W \subseteq X$ is an irreducible component, then the local ring $\mathcal{O}_{X,W}$ of X along W is artinian. Therefore, its length $l(\mathcal{O}_{X,W})$ is finite. The fundamental cycle of X is defined by

$$[X] = \sum_W l(\mathcal{O}_{X,W}) W, \tag{1.1}$$

where W runs over the set of irreducible components of X.

1.2. **Rational equivalence.** Let V be a variety and $W \subset V$ a subvariety of codimension one. Set $A = \mathcal{O}_{V,W}$, the stalk of the structure sheaf $\mathcal{O}_V$ at W. Thus, if $U = \mathrm{Spec}(R) \subset V$ is any affine open subset with non empty intersection with W, the ring $\mathcal{O}_{V,W}$ is just the localization $R_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal corresponding to the subvariety $W \cap U$ of U.

Denote by $R(V)$ the field of rational functions on V.

1.2.1. Definition. *Let $r \in R(V)$ be a non-zero rational function.*

The **order** *of r along a subvariety $W \subset V$ of codimension one is defined by the formula*

$$\mathrm{ord}_W(r) = l(A/\langle a \rangle) - l(A/\langle b \rangle)$$

where $A = \mathcal{O}_{V,W}$, and $r = a/b$, with $a, b \in A$. Here $l(M)$ is the length of the module M over A.

It can be shown that $\mathrm{ord}_W(r)$ does not depend on the representation $r = a/b$, and that it is furthermore additive:

$$\mathrm{ord}_W(rr') = \mathrm{ord}_W(r) + \mathrm{ord}_W(r') \quad \forall r, r' \in R(V)^*.$$

1.2.2. Definition. *The* **cycle of a rational function** *$r \in R(V)^*$ is defined by*

$$[r] = \sum_W \mathrm{ord}_W(r) \cdot W$$

where the sum extends over the collection of closed subvarieties of codimension one of V.

One checks that the sum is finite, *i.e.*, $\mathrm{ord}_W(r) = 0$ except for finitely many W's.

1.2.3. Example. *An obvious example is to consider the function $f(t) = t^n$ in $k[t]$. If $p \in \mathbb{A}^1$ is a point, we have $A = k[t]_{(m_p)} = k[t]_{(t-p)}$. In these case $A/\langle t^n \rangle = 0$ if $p \neq 0$ (because t is invertible in A) and $A/\langle t^n \rangle \simeq k \oplus kt \oplus \cdots \oplus kt^{n-1}$ if $p = 0$. Therefore*

$$\mathrm{ord}_0(t^n) = l(A/\langle t^n \rangle) = n$$

as expected.

The cycle of t^n is

$$[t^n] = \sum_p \mathrm{ord}_p(t^n) \cdot p = n.0\ .$$

1.2.4. Definition. *Let X be a scheme. The* **group of k-cycles rationally equivalent to zero** *is the subgroup $\mathcal{R}_k X \subset \mathcal{C}_k X$ generated by the cycles of rational functions of closed subvarieties of X of dimension $k+1$.*

The **Chow group** *of X is the graded group,*

$$\mathcal{A}_* X = \bigoplus_k \mathcal{A}_k X = \bigoplus_k \mathcal{C}_k X / \mathcal{R}_k X$$

Two cycles are said to be **rationally equivalent** *to each other if they represent the same class modulo $\mathcal{R}_* X$.*

1.2.5. Remark.
If X is of pure dimension n, then $\mathcal{A}_i X = 0$ for $i < 0$ and for $i > n$. Moreover, $\mathcal{A}_n X = \mathcal{C}_n X$ is the free abelian group generated by the irreducible components of X. If X is a variety of dimension n, we have $\mathcal{A}_n X = \mathcal{C}_n X = \mathbb{Z}$.

1.2.6. Examples.

(1) $\mathcal{A}_{n-1}\mathbb{A}^n = 0$, because any divisor in $\mathbb{A}^n$ is the zero locus of a polynomial (*i.e.,* a function in $\mathbb{A}^n$).
(2) $\mathcal{A}_0\mathbb{A}^n = 0$ for $n > 0$. In fact, if $P \in \mathbb{A}^n$, we can choose a linear function r on a line through P that vanishes exactly at P. Therefore $[r] = [P]$, and $[P] \in \mathcal{R}_0\mathbb{A}^n$.
(3) $\mathcal{A}_k\mathbb{A}^n = 0$ for $k < n$, see Proposition (1.5.3, p. 43).
(4) $\mathcal{A}_{n-1}\mathbb{P}^n = \mathbb{Z} \cdot h$, the free abelian group generated by h, the class of a hyperplane. Indeed, let F_d be a homogeneous polynomial of degree d and let $\mathcal{Z}(F_d)$ be the corresponding hypersurface. Let $[\mathcal{Z}(F_d)]$ denote its fundamental cycle (1.1). Consider the rational function $r := F_d/F_1^d \in R(\mathbb{P}^n)$. Then $[r] = [\mathcal{Z}(F_d)] - d\cdot[\mathcal{Z}(F_1)]$, *i.e.,* $[\mathcal{Z}(F_d)] \sim d\cdot[\mathcal{Z}(F_1)]$. Since $h = [\mathcal{Z}(F_1)]$ clearly is not torsion, $\mathcal{A}_{n-1}\mathbb{P}^n$ is freely generated by h.
(5) In general, for $0 \le k \le n$, $\mathcal{A}_k\mathbb{P}^n = \mathbb{Z}\cdot[\mathbb{P}^k]$, the free abelian group generated by the class of a linear subspace of dimension k, see Proposition 1.5.3.
(6) Let X be a smooth projective curve. Then we have

$$\mathcal{C}_0 X = \mathrm{Div}(X) \quad \text{and} \quad \mathcal{A}_0 X = \mathrm{Pic}(X)\,.$$

(7) If X has pure dimension n, an element of $\mathcal{C}_{n-1}(X)$ is a Weil divisor, and the quotient group $\mathcal{A}_{n-1}(X)$ is the group of Weil divisor classes. In this sense the Chow groups can be viewed as a generalization of Weil divisors classes.

1.3. **Direct Image.** Given a morphism $f : X \to Y$ of schemes, we shall define a natural homomorphism of groups $f_* : \mathcal{C}_* X \to \mathcal{C}_* Y$. Moreover, if f is *proper* then $f_*(\mathcal{R}_* X) \subset \mathcal{R}_* Y$, thereby inducing a homomorphism $\mathcal{A}_* X \to \mathcal{A}_* Y$.

1.3.1. Definition. *A morphism $f : X \to Y$ is* **proper** *if it is separated and universally closed,* i.e., *for all $Z \to Y$, the morphism induced by fiber product,*

$$X \times_Y Z \to Z$$

takes closed sets to closed sets.

A scheme X is **complete** *if the structural morphism $X \to \mathrm{Spec}(\mathbb{C})$ is proper.*

Properness corresponds to compact fibers in the classical topology, while completeness is the algebraic translation of compactness in the classical topology. For another characterization and properties of proper morphism consult [31].

We will use proper morphism in order to guarantee that the image of a variety is a closed subset.

1.3.2. Definition. *Let $f : V \to W$ be a dominant morphism of varieties. We define the* **degree** *of f as*

$$\deg(f) = \begin{cases} 0 & \textit{if } \dim V > \dim W \\ [R(V) : R(W)] & \textit{if } \dim V = \dim W \end{cases}$$

Observe that in the case $\dim V = \dim W$, the function fields $R(V)$ and $R(W)$ are finitely generated field extensions with the same transcendence degree over $\mathbb{C}$. Hence $\deg(f)$ is finite.

We can interpret the $\deg(f)$ as the degree of the covering or as the number of points in a fiber of f.

1.3.3. Definition. *Let $p : X \to Y$ be a proper map of schemes. Let $V \subset X$ be a closed subvariety and $W := p(V)$. Let $f : V \to W$ be the map induced by p. We put*

$$p_*(V) = \deg(f) \cdot W \in \mathcal{C}_* W.$$

We extend it by linearity to a homomorphism

$$p_* : \mathcal{C}_* X \to \mathcal{C}_* Y,$$

called **direct image** *of p.*

1.3.4. Examples.

(1) If X is complete, then any morphism $f : X \to Y$ is proper. Examples of complete varieties are $\mathbb{P}^n$, $\mathbb{G}(k,n)$, and any projective bundle $\mathbb{P}(\mathcal{E})$ associated to a vector bundle $\mathcal{E}$ over a complete base. See p. 83

These are in fact the varieties which we will use in the text.

(2) $\mathbb{A}^1$ is not complete. Indeed, the map $p : \mathbb{A}^1 \to pt$ is not proper: consider $\hat{p} : \mathbb{A}^2 \to \mathbb{A}^1$, $\hat{p}(x,y) = x$. Then $\hat{p}(\{xy = 1\}) = \mathbb{A}^1 \backslash \{0\}$ that is not closed. In this case there are no direct image map, because $\mathcal{A}_0(\mathbb{A}^1) = 0 \to \mathcal{A}_0(pt) = \mathbb{Z}$, i.e. the class of a point is zero in $\mathcal{A}_0(\mathbb{A}^1)$ but non zero in $\mathcal{A}_0(pt)$.

(3) The same occurs for the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. This map is not proper, and there are no direct image, because the class of a point is zero in $\mathcal{A}_0(\mathbb{A}^1)$ but non zero in $\mathcal{A}_0(\mathbb{P}^1) = \mathbb{Z}$.

1.3.5. Theorem. *Let $p : X \to Y$ be a proper map. Then the direct image map $p_* : \mathcal{C}_* X \to \mathcal{C}_* Y$ preserves rational equivalence, i.e, we have $p_*(\mathcal{R}_* X) \subset \mathcal{R}_* Y$.*

Proof. See [21, Theorem 1.4. p. 11]. □

Using this result we can define the direct image of a morphism at the Chow group level.

1.3.6. Definition. *Let $p : X \to Y$ be a proper map. The* **direct image homomorphism** *is the induced homomorphism*

$$p_* : \mathcal{A}_* X \to \mathcal{A}_* Y.$$

1.3.7. Definition. *Let X be a complete scheme, and $f : X \to \mathrm{pt}$ be the natural proper map. For any $0-$cycle $\alpha \in \mathcal{A}_0(X)$ we define the degree of α to be $f_*(\alpha) \in \mathcal{A}_0(\mathrm{pt}) = \mathbb{Z}$. We write*

$$\deg(\alpha) = \int_X \alpha\,.$$

The degree is the number of points counted with the appropriate multiplicity.

1.3.8. Proposition. *The direct images are functorial,* i.e., *if $f : X \to Y$ and $g : Y \to Z$ are proper maps, then $(g \circ f)_* = g_* \circ f_*$. In particular, if $p : X \to Y$ is a proper map of complete schemes, we have*

$$\int_X \alpha = \int_Y p_*(\alpha), \forall \alpha \in \mathcal{A}_0(X).$$

1.4. **Inverse Image.** Let $f : X \to Y$ be a *flat* morphism (cf. [31]). We shall define a homomorphism $f^* : \mathcal{C}_*(Y) \to \mathcal{C}_*(X)$ that preserves rational equivalence, *i.e.,* $f^*(\mathcal{R}_*(Y)) \subset \mathcal{R}_*(X)$, and therefore induces a homomorphism

$$f^* : \mathcal{A}_*(Y) \to \mathcal{A}_*(X).$$

1.4.1. Definition. *Let $V \subset Y$ be a subvariety. The* **inverse image cycle** *of V under f is defined by*

$$f^*V = [f^{-1}(V)].$$

The right hand side above is the fundamental cycle ((1.1, p. 39)) of the closed subscheme $f^{-1}(V) \subseteq X$. We extend it by linearity to obtain a homomorphism

$$f^* : \mathcal{C}_*(Y) \to \mathcal{C}_*(X).$$

We say that $f : X \to Y$ is of **relative dimension** n if for each subvariety W of Y, any component V of $f^{-1}(W)$ is of dimension

$$\dim V = n + \dim W.$$

It is the case of any fibration, and these will be the morphisms that we will work with in the text.

A flat morphism will be assumed to have relative dimension n for some n. We register the following

1.4.2. Proposition. *Let $f : X \to Y$ be a flat morphism of relative dimension n. Then for each closed subscheme $Z \subseteq Y$ of pure dimension k, we have*

$$f^*[Z] = [f^{-1}Z] \quad \text{in } \mathcal{C}_{k+n}X.$$

We list out the principal examples of flat morphisms that occur in this text.

1.4.3. Examples.

(1) Any open imbedding.
(2) The structure map of a vector bundle, or a projective bundle to its base.
(3) The projection $X \times Y \to X$ where Y is a pure dimensional scheme.

Flat families (fibers of flat morphisms) are the adequate notion to work with families of schemes, for example, for flat families of subschemes in $\mathbb{P}^n$ the fibers have constant Hilbert polynomial (this mean that numerical invariants as dimension, degree etc, are preserved)

1.4.4. Proposition. *Let $f : X \to Y$ be a flat morphism of relative dimension n. Then $f^* : \mathcal{C}_kY \to \mathcal{C}_{k+n}X$ and $f^*(\mathcal{R}_kY) \subset \mathcal{R}_{k+n}X$. Therefore we obtain a homomorphism*

$$f^* : \mathcal{A}_kY \to \mathcal{A}_{k+n}X.$$

The inverse image is functorial. By this we mean that, if $f : X \to Y$, and $g : Y \to Z$ are flat morphisms of schemes, then

$$(g \circ f)^* = f^* \circ g^*.$$

Proof. See [21, Theorem 1.7, p. 19]. □

The inverse image is compatible with proper direct images:

1.4.5. Proposition. *Let be given a Cartesian diagram,*

$$\begin{tikzcd} X \times_Y Y' \arrow[r, equal] & :X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

where f is flat of relative dimension n and g is proper. Then f' (resp. g') *is flat of relative dimension n* (resp. proper) *and we have*

$$g'_* f'^* = f^* g_* : \mathcal{C}_k Y' \longrightarrow \mathcal{C}_{k+n} X.$$

1.5. **Excision.** The following proposition is a very useful tool that allows us to compute the Chow groups of $\mathbb{A}^n$ and $\mathbb{P}^n$.

1.5.1. Proposition. *Let $i : Z \hookrightarrow X$, $j : U \hookrightarrow X$ the inclusion maps of a closed subscheme Z and its complement U. Then we have the following exact sequence:*

$$\mathcal{A}_* Z \xrightarrow{i_*} \mathcal{A}_* X \xrightarrow{j^*} \mathcal{A}_* U \to 0.$$

Proof. □

1.5.2. Lemma. *Let X be a scheme and let $p : X \times \mathbb{A}^n \to X$ be the projection. Then*

$$p^* : \mathcal{A}_* X \longrightarrow \mathcal{A}_*(X \times \mathbb{A}^n)$$

is surjective.

Proof. See [21, Proposition 1.9, p. 22]. □

Using these two results we can compute the Chow groups of $\mathbb{A}^n$ and $\mathbb{P}^n$.

1.5.3. Proposition.

(1) $\mathcal{A}_i \mathbb{A}^n = 0$ *for all $i \neq n$, and* $\mathcal{A}_n \mathbb{A}^n = \mathbb{Z}$.
(2) $\mathcal{A}_i \mathbb{P}^n = \mathbb{Z}[\mathbb{P}^i]$, *the free group generated by the class of a dimension i subspace $\mathbb{P}^i \subset \mathbb{P}^n$ for all $0 \leq i \leq n$.*

Proof. We already know (1.2.6, p. 40) that $\mathcal{A}_n \mathbb{A}^n = \mathbb{Z}$ and $\mathcal{A}_{n-1} \mathbb{A}^n = 0$. If $i < n-1$, by the lemma above we have a surjective map $\mathcal{A}_{i-n+1} \mathbb{A}^1 \twoheadrightarrow \mathcal{A}_i(\mathbb{A}^1 \times \mathbb{A}^{n-1})$. But $\mathcal{A}_{i-n+1} \mathbb{A}^1 = 0$ for $i < n-1$. This proves (1).

We prove (2) using induction and the excision sequence

$$\mathcal{A}_i \mathbb{P}^{n-1} \longrightarrow \mathcal{A}_i \mathbb{P}^n \longrightarrow \mathcal{A}_i \mathbb{A}^n = 0.$$

By induction $\mathcal{A}_i \mathbb{P}^{n-1} = \mathbb{Z}[\mathbb{P}^i]$. Hence $\mathcal{A}_i \mathbb{P}^n$ is generated by $[\mathbb{P}^i]$. It remains to prove that $m[\mathbb{P}^i] = 0$ in $\mathcal{A}_i \mathbb{P}^n$ implies $m = 0$. Suppose

$$m[\mathbb{P}^i] = \sum m_k [r_k]$$

for some integers m_k and some rational functions $r_k \in R(V_k)$, where V_k's are subvarieties of $\mathbb{P}^n$ of dimension $i+1$. Set $Z := \bigcup V_k$, then $m[\mathbb{P}^i] = 0$ in $\mathcal{A}_i Z$. There exists a finite map $p : Z \to \mathbb{P}^{i+1}$ (e.g., induced by a linear projection). We find

$$m p_* [\mathbb{P}^i] = 0 \text{ in } \mathcal{A}_i \mathbb{P}^{i+1}$$

which is torsion free. Hence $m = 0$ as desired. □

1.5.4. Definition. *Let* $\alpha = \sum_{i=0}^{k} m_i \cdot [\mathbb{P}^i]$ *be a cycle in* $\mathbb{P}^n$. *If* $m_k \neq 0$ *we define the* **degree** *of* α *by the formula*

$$\deg(\alpha) = m_k.$$

1.6. **Chern classes.** In this section we define Chern classes associated to a vector bundle $\mathcal{E}$ over a scheme X (cf. Appendix A.1). These classes are constructed as operators on the Chow groups $\mathcal{A}_*X$. See [21, Chapter 3].

1.6.1. Definition. *Let* X *be a scheme, and* $\mathcal{E}$ *a vector bundle over* X *of rank* e. *The* $i-th$ *Chern class of* $\mathcal{E}$ *is a homomorphism*

$$c_i(\mathcal{E}) \cap \_ : \mathcal{A}_k X \to \mathcal{A}_{k-i} X$$

characterized by the following five properties:

(1) $c_0(\mathcal{E}) = 1$ (= identity operator).

(2) (Naturality) *If* $f : Y \to X$ *is a flat morphism, then*

$$f^*(c_i(\mathcal{E}) \cap \alpha) = c_i(f^*\mathcal{E}) \cap f^*\alpha$$

for all cycle $\alpha \in \mathcal{A}_*X$ *and all* i. *Here* $f^*\mathcal{E}$ *is the pull-back of* $\mathcal{E}$ *by* f (cf. Appendix A.1.3).

(3) (Whitney sum) *If*

$$0 \to \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \to 0$$

is an exact sequence of vector bundles, then

$$c_i(\mathcal{E}) = \sum_{r+s=i} c_r(\mathcal{E}')c_s(\mathcal{E}'').$$

(4) (Normalization) *If* $\mathcal{E}$ *is a line bundle, and* D *is a Cartier divisor on* X *such that* $\mathcal{O}_X(D) \simeq \mathcal{E}$ *(see Appendix A.2), then*

$$c_1(\mathcal{E}) \cap [X] = [D].$$

(5) (Projection formula) *If* $f : Y \to X$ *is a proper morphism, then*

$$f_*(c_i(f^*\mathcal{E}) \cap \alpha) = c_i(\mathcal{E}) \cap f_*\alpha$$

for all $\alpha \in \mathcal{A}_*Y$.

1.6.2. Remarks. (1) Since $\mathcal{A}_i(X) = 0$ for $i < 0$, we see that $c_i(\mathcal{E})$ is nilpotent.
(2) It is a fundamental (and nontrivial) fact that if $\mathcal{E}, \mathcal{E}'$ are vector bundles, then the operators $c_i(\mathcal{E})$ and $c_j(\mathcal{E}')$ commute.
(3) We will also see that $c_i(\mathcal{E}) = 0$ if $i > e = \mathrm{rk}(\mathcal{E})$.

1.6.3. Definition. *Let* $\mathcal{E}$ *be a vector bundle over a scheme* X.
The **total Chern class** *of* $\mathcal{E}$ *is*

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots$$

By the above remark, we see that this sum is finite and $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots$ is an invertible element of the endomorphism ring of $\mathcal{A}_*X$.

We define the **total Segre class** of $\mathcal{E}$ as the formal inverse of $c(\mathcal{E})$,

$$s(\mathcal{E}) = c(\mathcal{E})^{-1} = 1 + s_1(\mathcal{E}) + \cdots .$$

1.6.4. Remark. Expanding $s = 1 + s_1 + \cdots = 1/(1 + c_1 + \cdots)$ we find $s_1 = -c_1, s_2 = c_1^2 - c_2$, etc. Each $s_i(\mathcal{E})$ defines a homomorphism $\mathcal{A}_k X \to \mathcal{A}_{k-i} X$. It can be proved that

$$s_i(\mathcal{E}) \cap \alpha = p_*(c_1(\mathcal{O}_{\mathcal{E}}(1))^{e-1+i} \cap p^*\alpha)$$

for $\alpha \in \mathcal{A}_k X$, where $e = \mathrm{rk}(\mathcal{E})$ and $p : \mathbb{P}(\mathcal{E}) \to X$ is the projection (see Appendix A.3).

In fact the formula can be taken as the definition of Segre class.(cf. [21])

Recall the definition of degree of a cycle (Definition 1.5.4). We have that if $X \subset \mathbb{P}^n$ is a subscheme of pure dimension k and degree d then

$$d = \deg(h^k \cap [X])$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

As an application of the definition and properties of Chern and Segre classes, we shall prove a lemma that will be a very usefull tool in the sequel.

1.6.5. Lemma. *Let V be a vector space of dimension $N+1$. Let $\mathcal{E}$ be a subbundle of the trivial bundle $X \times V$ over a variety X of dimension n. Consider $\mathbb{P}(\mathcal{E})$ the projective bundle associated to $\mathcal{E}$, and $\mathbb{P}^N = \mathbb{P}(V)$. Consider the following diagram:*

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}) \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X & & \mathbb{P}^N \end{tikzcd} .$$

Let $M \subseteq \mathbb{P}^N$ denote the image of q_2. Suppose that q_2 is generically finite. Then

$$\deg(q_2) \deg M = \int s_n(\mathcal{E}) \cap [X].$$

Proof. Set for short $\delta = \deg(q_2)$. Hence $q_{2*}[\mathbb{P}(\mathcal{E})] = \delta[M]$. Now we have

$$\begin{aligned} \deg M = \int H^\nu \cap [M] &= \frac{1}{\delta} \int H^\nu \cap q_{2*}[\mathbb{P}(\mathcal{E})] \\ &= \frac{1}{\delta} \int q_2^* H^\nu \cap [\mathbb{P}(\mathcal{E})] = \frac{1}{\delta} \int \widetilde{H}^\nu \cap [\mathbb{P}(\mathcal{E})] \end{aligned}$$

where $\nu = \dim(\mathbb{P}(\mathcal{E})) = \dim M$, $H = c_1(\mathcal{O}_{\mathbb{P}^N}(1))$, $\widetilde{H} = c_1(\mathcal{O}_{\mathcal{E}}(1))$.

Set $e = \mathrm{rk}(\mathcal{E})$. Thus $\nu = e - 1 + n$. Hence

$$\int \widetilde{H}^\nu \cap [\mathbb{P}(\mathcal{E})] = \int q_{1*}(\widetilde{H}^\nu \cap q_1^*[X]) = \int s_n(\mathcal{E}) \cap [X]$$

by Remark 1.6.4. The proof is complete. □

The following Lemma will be used repeatedly in order to prove the generic injectivity of certain maps.

1.6.6. Lemma. *In the situation of the previous lemma, in order to prove that q_2 is generically one to one, it suffices to find a point $[v] \in M$ such that the fiber $q_2^{-1}([v])$ consists of one reduced point.*

Proof. If we prove the existence of such point $[v]$, by the theorem on the dimension of fibers (see [43, Chapter I §6.3]) there exists an open set U in M such that the fiber over each point in U has dimension zero, ($U \neq \emptyset$ because $[v] \in U$). Therefore q_2 is generically finite. Shrinking U we may assume (i) U is affine, say with coordinate ring A and (ii) the restriction of q_2 over ${q_2}^{-1}U$ is finite (cf. [31, ex.11.2, p. 280]). It follows that ${q_2}^{-1}U$ is affine, with coordinate ring B which is an A-module of finite type. Now for each point $u \in U$ with corresponding maximal ideal $m_u \subset A$, the fiber ${q_2}^{-1}u = \operatorname{Spec}(B/m_uB)$ is finite and consists of $\dim_{\mathbb{C}}(B/m_uB)$ points counted with multiplicity. By semicontinuity, this vector space dimension attains a minimum over an open subset. Since the fiber over $y = [v]$ consists of one reduced point, that mininum is precisely one and we are done. □

Notice that reducedness of ${q_2}^{-1}([v])$ required above means that the tangent map of q_2 is injective.

Chern classes and Segre classes of vector bundles can be effectively computed with appropriate tools. The principal one is the splitting principle that we state in the following proposition. This, together with the next lemma give us a handy way to compute Chern classes.

1.6.7. Proposition. (The splitting principle) *Let $\mathcal{E}$ be a vector bundle over a scheme X. Then there exists a flat map $f : X' \to X$ such that*

(1) *the induced homomorphism $f^* : \mathcal{A}_*X \to \mathcal{A}_*X'$ is injective.*
(2) *$f^*\mathcal{E}$ admits a filtration by vector subbundles*

$$\mathcal{E}_e = 0 \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = f^*\mathcal{E}$$

whose successive quotients are line bundles, $L_i = \mathcal{E}_{i-1}/\mathcal{E}_i$.

Proof. See [21, § 3.2.]. □

1.6.8. Lemma. *Let $\mathcal{E}$ be a vector bundle endowed with a filtration as in the proposition above. Put $\lambda_i = c_1(L_i)$ and define*

$$\begin{cases} \sigma_1 = \sum_i \lambda_i, \\ \sigma_2 = \sum_{i<j} \lambda_i\lambda_j, \\ \quad \vdots \\ \sigma_e = \lambda_1 \cdots \lambda_e, \end{cases}$$

the elementary symmetric functions. Then we have

$$c(\mathcal{E}) = \Pi_1^e(1 + \lambda_i), \quad \textit{that is,}$$

$$c_i(\mathcal{E}) = \begin{cases} \sigma_i & \textit{for } 1 \le i \le e; \\ 0 & \textit{for } i > e. \end{cases}$$

Proof. See [21, Remark 3.2.3. p. 54.]. □

Using the splitting principle, we see that in order to show formulas involving Chern classes we can suppose that the vector bundle $\mathcal{E}$ has a filtration with line bundle quotients. In fact, the Chern classes of $\mathcal{E}$ are the same as of $\bigoplus L_i$. The classes $\lambda_i = c_1(L_i)$ are the **Chern roots** of $\mathcal{E}$. The Chern classes are the symmetric elementary functions of the chern roots $\{\lambda_i\}$.

1.6.9. Proposition.

(1) Dual bundles. *The Chern classes of the dual bundle $\mathcal{E}^\vee$ are given by the formula*

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E}).$$

(2) Twisted bundles. *Suppose that L is a line bundle, and* $\mathrm{rk}(\mathcal{E}) = e$, *then*

$$c_e(\mathcal{E} \otimes L) = \sum_{i=0}^{e} c_1(L)^i c_{e-i}(\mathcal{E}).$$

□

The following proposition is the key to many interesting geometric applications of Chern classes.

Let $\mathcal{E}$ be a vector bundle of rank e over a scheme X. We say a section s of $\mathcal{E} \to X$ is *regular* at a point $x \in X$ if there is a local trivialization of $\mathcal{E}$ around x such that s is given by $(s_1, \ldots, s_e)$ where either some s_i is a unit or the $s_i \in \mathcal{O}_{X,x}$ form a regular sequence. This means that s_1 is a nonzero divisor at the stalk $\mathcal{O}_{X,x}$ and each s_i is a nonzero divisor in $\mathcal{O}_{X,x}/\langle s_1, \ldots, s_{i-1}\rangle$. We say s is regular if it is so at each point. If X is a smooth variety, regularity of s is tantamount to requiring the scheme of zeros $\mathcal{Z}(s)$ to be either empty or of the correct codimension $e = \mathrm{rk}\,\mathcal{E}$.

1.6.10. Proposition. *Let $\mathcal{E}$ be a vector bundle of rank e over a scheme X of pure dimension n. Let s be a regular section of $\mathcal{E}$. Then*

$$c_e(\mathcal{E}) \cap [X] = [\mathcal{Z}(s)] \text{ in } \mathcal{A}_{n-e}X.$$

1.6.11. Example Let $\mathcal{X}$ be a vector field in $\mathbb{P}^n$ that defines a regular section of $\mathcal{T}\mathbb{P}^n$. Then $\mathcal{Z}(\mathcal{X})$ = scheme of singularities of $\mathcal{X}$, is a finite set. To compute the number of singularities we can use the proposition to obtain

$$\int c_n(\mathcal{T}\mathbb{P}^n) \cap [\mathbb{P}^n] = \int [\mathcal{Z}(\mathcal{X})].$$

We shall return to this in the next Section.

1.7. **Some Chow groups.** In this section we give explicit description of the Chow groups of projective vector bundles (in particular for $\mathbb{P}^n$) and Grassmannians.

1.7.1. Proposition. *Let $\mathcal{E}$ be a vector bundle of rank e over a scheme X. Then*

$$\mathcal{A}_*\mathbb{P}(\mathcal{E}) \simeq \frac{\mathcal{A}_*(X)[H]}{\langle H^e + c_1(\mathcal{E})H^{e-1} + \cdots + c_{e-1}(\mathcal{E})H + c_e(\mathcal{E})\rangle}$$

where $H = c_1(\mathcal{O}_{\mathcal{E}}(1))$, see Appendix A.3.

In particular we have, for $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$

$$\mathcal{A}_*\mathbb{P}^n \simeq \mathbb{Z}[h]/\langle h^{n+1}\rangle$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. See [21, Ex. 8.3.4, p. 141], [46, Ch. 10].. □

1.7.2. Proposition. *Let $\mathbb{G}(k, n)$ the Grassmannian of k-planes of $\mathbb{P}^n$. We have*

$$\mathcal{A}_*\mathbb{G}(k, n) \simeq \mathbb{Z}[a, b]/\langle a_*b_* - 1\rangle$$

where $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_{n-k})$ are indeterminates, $a_* = 1 + a_1 + \cdots + a_k$, $b_* = 1 + b_1 + \cdots + b_{n-k}$.

Proof. See [46, Ch. 10]. □

Exercise 1. Write explicitly the above relations for $\mathbb{G}(1, 3)$.

2. Holomorphic Foliations in Projective Spaces

In this Section we introduce the basic notions of foliations of dimension one (respectively, codimension one) in $\mathbb{P}^n$. We review the definitions of degree, singularity, order of a singularity and invariant subvarieties, in the way that will be used in the text.

Throughout this work S_d will denote the vector space of homogeneous polynomials of degree d in the $(n+1)$ homogeneous coordinates $Z_0, \dots, Z_n$. We have

$$S_d = \mathrm{Sym}_d\, \check{\mathbb{C}}^{n+1}\ ; \quad \dim S_d = \binom{d+n}{n}.$$

We will identify

$$S_1 = \check{\mathbb{C}}^{n+1} \simeq \Omega_0 \mathbb{C}^{n+1}$$

the vector space with basis

$$\{dZ_0, \dots, dZ_n\}\,.$$

Similarly, we will identify

$$S_1^\vee = \mathbb{C}^{n+1} \simeq \mathcal{T}_0 \mathbb{C}^{n+1} \tag{2.1}$$

the vector space with basis

$$\left\{ \frac{\partial}{\partial Z_0}, \dots, \frac{\partial}{\partial Z_n} \right\}.$$

2.1. **Tangent and cotangent bundles of $\mathbb{P}^n$.** The tangent bundle of $\mathbb{P}^n$ is determined by the Euler exact sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathbb{C}^{n+1} \longrightarrow \mathcal{T}\mathbb{P}^n \to 0. \tag{2.2}$$

The first map is

$$1 \mapsto (Z_0, \dots, Z_n),$$

The second map is defined by

$$F = (F_0, \dots, F_n) \mapsto \delta_F,$$

where δ_F is the derivation $\delta_F(f/g) = \frac{f\nabla g - g \nabla f}{g^2} \cdot (F_0, \dots, F_n)$.

Dualizing the Euler sequence we have

$$0 \to \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \otimes \check{\mathbb{C}}^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \to 0. \tag{2.3}$$

The rightmost map is given by $(f_0, \dots, f_n) = \sum f_i dZ_i \mapsto \sum_{i=0}^n f_i Z_i$. Recall that the sections of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over an open subset $U \subset \mathbb{P}^n$ are given by fractions F/G such that F, G are homogeneous polynomials of degrees $\deg F = \deg G - 1$ and G has no zeros over U. Thus each $f_i Z_i$ in the sum is of degree zero, *i.e.,* a function.

2.2. **Dimension one foliations.**

2.2.1. Definition. *A* **dimension one foliation** *in $\mathbb{P}^n$ is a nonzero global section of $\mathcal{T}\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d-1)$ for some $d \geq 0$ modulo non-zero complex multiples.*

Let us denote

$$V_{1,n,d} = H^0(\mathbb{P}^n, \mathcal{T}\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d-1)) \tag{2.4}$$

and

$$\mathbb{F}(1, n, d) = \mathbb{P}(V_{1,n,d}).$$

Then a dimension one foliation is an element $\mathcal{X} \in \mathbb{F}(1, n, d)$.

Tensoring the Euler sequence by $\mathcal{O}_{\mathbb{P}^n}(d-1)$ we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d-1) \to \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathbb{C}^{n+1} \to \mathcal{T}\mathbb{P}^n(d-1) \to 0. \tag{2.5}$$

Taking global sections in the last sequence and using that $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) = 0$ ([31, Chapter III, Theorem 5.1]) we obtain the following exact sequence:

$$0 \to S_{d-1} \to S_d \otimes S_1^\vee \to H^0(\mathbb{P}^n, \mathcal{T}\mathbb{P}^n(d-1)) \to 0.$$

From this we deduce that a foliation is given in homogeneous coordinates by a vector field

$$X = F_0 \frac{\partial}{\partial Z_0} + \cdots + F_n \frac{\partial}{\partial Z_n},$$

where F_i are homogeneous polynomials of degree d, modulo multiples of the radial vector field

$$R := Z_0 \frac{\partial}{\partial Z_0} + \cdots + Z_n \frac{\partial}{\partial Z_n}.$$

We will denote by X an element in $S_d \otimes S_1^\vee$, and $\mathcal{X} := X$ modulo $S_{d-1} \cdot R$.

The Euler sequence gives us:

$$V_{1,n,d} \simeq \frac{S_d \otimes S_1^\vee}{S_{d-1} \cdot R} \tag{2.6}$$

From this it is clear that

$$N_{1,n,d} := \dim V_{1,n,d} - 1 = (n+1)\binom{d+n}{n} - \binom{d-1+n}{n} - 1 \tag{2.7}$$

and

$$\mathbb{F}(1,n,d) = \mathbb{P}^{N_{1,n,d}}.$$

2.2.2. Definition. The **degree of a foliation** $\mathcal{X} \in \mathbb{F}(1,n,d)$ is d.

For a geometric interpretation of the degree, take a hyperplane H in $\mathbb{P}^n$. Define

$$\mathcal{T}(\mathcal{X}, H) = \{p \in H \mid \mathcal{X}(H)(p) = 0\},$$

the **set of tangencies** of $\mathcal{X}$ with H. For a generic H, it can be seen that $\mathcal{T}(\mathcal{X}, H)$ has codimension one in H and the degree of $\mathcal{X}$ is the degree of $\mathcal{T}(\mathcal{X}, H)$ ([39, Chapter II §3]).

In fact, if we take a hyperplane defined by the equation

$$H := a_0 Z_0 + \cdots + a_n Z_n = 0$$

then $\mathcal{T}(\mathcal{X}, H)$ is given in H by

$$\mathcal{X}(H) := a_0 F_0 + \cdots + a_n F_n = 0.$$

For H generic, the polynomial $\mathcal{X}(H)$ is not identically zero and has degree d.

Observe that in $\mathbb{P}^2$ the set of tangencies of a degree d vector field $\mathcal{X}$ with a generic line is finite and consists of d points.

Exercise 2. The goal is to deduce the local expression of a vector field, using the Euler sequence. Let

$$\mathcal{X} = F_0 \frac{\partial}{\partial Z_0} + \cdots + F_n \frac{\partial}{\partial Z_n}$$

be a degree d vector field. Set

$$U_j := \{[Z_0 : \cdots : Z_n] \mid Z_j \neq 0\} = \{(z_0, \ldots, \hat{z_j}, \ldots, z_n) \in \mathbb{C}^n\}.$$

Prove that we have the following local expression for $\mathcal{X}$ on U_j:

$$\mathcal{X}_{U_j} = \sum_{i\neq j} a_i \frac{\partial}{\partial z_i} = \sum_{i\neq j} (f_i - z_i f_j)\frac{\partial}{\partial z_i} = \sum_{i\neq j} f_i \frac{\partial}{\partial z_i} - f_j \sum_{i=1} z_i \frac{\partial}{\partial z_i} \tag{2.8}$$

where f_i is the dehomogenization of F_i with respect to Z_j.

The notation U_j for the canonical open set $Z_j \neq 0$ will be used in all the text.

2.2.3. Definition. *We say that $p \in \mathbb{P}^n$ is a* **singularity** *of $\mathcal{X}$ if p is a zero of the section*

$$\mathcal{X} : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{T}\mathbb{P}^n(d-1).$$

Explicitly, using the local expression in (2.8), the singularities of $\mathcal{X}$ in U_j are the common zeros of $\{a_i \mid i \neq j\}$. Alternatively, using homogeneous coordinates, the singularities of $\mathcal{X}$ are given by the ideal of $2\times 2-$minors, $Z_iF_j - Z_jF_i$ of the matrix $\left(\begin{smallmatrix} Z_0 & \cdots & Z_n \\ F_0 & \cdots & F_n \end{smallmatrix}\right)$, cf. [14], i.e. the singularities are the points p where $\mathcal{X}(p)$ has the same direction that $R(p)$.

Exercise 3. Using the Euler sequence we can compute the number of singularities of a generic vector field of degree d. The fact that $\mathcal{X}$ is generic implies that it has isolated singularities (see [32]). By Proposition (1.6.10, p. 47) we have:

$$\#\mathcal{Z}(\mathcal{X}) = c_n(\mathcal{T}\mathbb{P}^n(d-1)).$$

Prove that a dimension one foliation of degree d in $\mathbb{P}^n$ has

$$d^n + d^{n-1} + \cdots + d + 1$$

singularities (counting multiplicities). (Hint: use sequence (2.5, p. 49) and the properties of Chern classes.)

2.2.4. Definition. Order of a singularity.

Let p be a singularity of a vector field $\mathcal{X}$. Suppose that $p \in U_j$ and

$$\mathcal{X}_{U_j} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}$$

is the local expression of $\mathcal{X}$ in U_j. Then the **order** (sometimes named **algebraic multiplicity**) of the singularity p is

$$\nu_p(\mathcal{X}) = \min\{order_p(a_i) \mid i = 1, \ldots, n\}.$$

As usual, the order $order_p(a)$ of a polynomial a at a point p means the order of vanishing: $\min\{s \mid \partial^s a/\partial z_I(p) \neq 0, |I| = s\}$.

Exercise 4. Check that the order of a singularity is independent of the choice of the open set U_j such that $p \in U_j$.

2.2.5. Definition. Invariant hypersurface.

Let $Z \subset \mathbb{P}^n$ be an irreducible hypersurface defined by a homogeneous polynomial G of degree k, and $\mathcal{X}$ be a vector field of degree d. We say that Z **is invariant by** $\mathcal{X}$ if

$$\mathcal{X}(p) \in \mathcal{T}_p Z$$

for all $p \in Z \setminus (Sing(Z) \cup Sing(\mathcal{X}))$.

If Z is reducible, we say that it is invariant by $\mathcal{X}$ if and only if each irreducible component of Z is invariant by $\mathcal{X}$.

Exercise 5.

(1) Prove that if G is irreducible, the above condition is equivalent to the existence of a degree $d-1$ homogeneuos polynomial H such that

$$dG(\mathcal{X}) = \mathcal{X}(G) = GH.$$

Hint: Use the Hilbert's Nullstellensatz.

(2) Prove that this condition does not depend on the representative of $\mathcal{X}$ in $S_d \otimes S_1^\vee$.

Hint: By the Euler relation we have that $R(G) = k \cdot G$.

(3) If G is reducible, and

$$G = G_1^{r_1} \cdots G_n^{r_n}$$

is a decomposition of G into irreducible factors, prove that $\mathcal{X}(G) = HG$ for some H if and only if for all $i = 1, \ldots, n$ we have $\mathcal{X}(G_i) = H_i G_i$ for some H_i's.

2.2.6. Definition. Invariant algebraic subvariety.

If $Z \subset \mathbb{P}^n$ is an algebraic subvariety defined by the ideal $I_Z := \langle G_1, \ldots, G_r \rangle$ and $\mathcal{X}$ is a vector field, we say that Z **is invariant by** $\mathcal{X}$ if

$$\mathcal{X}(p) \in \mathcal{T}_p Z$$

for all $p \in Z \setminus (Sing(Z) \cup Sing(\mathcal{X}))$.

Exercise 6. If I_Z is saturated, this condition is equivalent to

$$dG_i(\mathcal{X}) = \mathcal{X}(G_i) \in I_Z \qquad \text{for all } i = 1, \ldots, r.$$

The hypothesis of the ideal to be saturated is necessary. For example $\mathcal{Z}(Z_0)$ is invariant by $\frac{\partial}{\partial Z_1}$, but $\frac{\partial}{\partial Z_1}(Z_0 Z_1)$ is not in the ideal $\langle Z_0^2, Z_0 Z_1, \ldots, Z_0 Z_n \rangle$.

2.3. **Codimension one foliations.** In this section we define codimension one foliations in $\mathbb{P}^n$ (*i.e.,* foliations defined by integrable one forms in $\mathbb{P}^n$). However, in the text we only deal with codimension one foliations in $\mathbb{P}^2$, so we discuss this case and the correspondence between vector fields and forms in $\mathbb{P}^2$. For further reading see [38].

2.3.1. Definition. A **projective one form** of degree d in $\mathbb{P}^n$ is given by a global section of $\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d+2)$, for some $d \geq 0$.

As in the case of vector fields in $\mathbb{P}^n$ we will deduce an expression in homogeneous coordinates for a form in $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d+2))$.

Tensoring the (dual of the) Euler sequence (2.2) by $\mathcal{O}_{\mathbb{P}^n}(d+2)$ we obtain

$$0 \to \Omega_{\mathbb{P}^n}(d+2) \to \mathcal{O}_{\mathbb{P}^n}(d+1) \otimes S_1 \to \mathcal{O}_{\mathbb{P}^n}(d+2) \to 0 \tag{2.9}$$

Taking global sections, and using that $H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2)) = 0$ we obtain the following exact sequence:

$$0 \to H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2)) \longrightarrow S_{d+1} \otimes S_1 \xrightarrow{\iota_R} S_{d+2} \to 0$$

where

$$\iota_R(\sum A_i dZ_i) = \sum A_i Z_i$$

is the contraction by the radial vector field. Hence $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2))$ is the kernel of ι_R. It follows that a one form $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2))$ can be written in homogeneous coordinates as

$$\omega = A_0 dZ_0 + \cdots + A_n dZ_n$$

where the A_i's are homogeneous polynomials of degree $d+1$ satisfying

$$A_0 Z_0 + \cdots + A_n Z_n = 0.$$

Notice that a one form induces a distribution of codimension one subspaces of $\mathcal{T}\mathbb{P}^n$ given by $p \mapsto \mathrm{Ker}\omega_p$. However, this distribution is not necessarily integrable. (A distribution is called **integrable** if there exists a smooth germ of hypersurface U at p such that $\mathcal{T}_q U = \mathrm{Ker}\omega_q$, $\forall\, q \in U$ near p).

The condition to be integrable is expressed by Frobenius equation,

$$\omega \wedge d\omega = 0.$$

For a geometric interpretation see [38] or [4].

2.3.2. Definition. A **codimension one foliation** is an integrable projective one form modulo non zero complex multiples.

Denote

$$V_{n-1,n,d} = H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2)).$$

A codimension one foliation is given by an element ω of

$$\mathbb{F}(n-1, n, d) := \mathbb{P}(V_{n-1,n,d})$$

such that $\omega \wedge d\omega = 0$.

2.3.3. Remark. The condition $\omega \wedge d\omega = 0$ translates into a system of quadratic equations in $\mathbb{P}(V_{n-1,n,d})$. They define the scheme of codimension one foliations of degree d in $\mathbb{P}^n$. Very little is known about it. The problem of determining their irreducible components remains a rather challenging field of research, cf. [38].

Next we explain the local expression of a projective one form.

Let U_j denote the affine open set $Z_j \neq 0$, with coordinates $(z_0, \ldots, \hat{z_j}, \ldots, z_n)$. The local expression of a one form

$$\omega = A_0 dZ_0 + A_1 dZ_1 + \cdots + A_n dZ_n$$

on U_j is

$$\omega_{U_j} = a_0 dz_0 + \cdots + \widehat{a_j dz_j} + \cdots + a_n dz_n$$

where a_i is the dehomogenization of A_i with respect to Z_j.

2.3.4. Definition. The **degree of a codimension one foliation** is d if it is given by a one form in $\mathbb{F}(n-1, n, d)$.

Geometrically, if $\omega \in \mathbb{F}(n-1, n, d)$, the degree is the number of tangencies of the distribution induced by ω with a generic line in $\mathbb{P}^n$. For example, suppose that ω is given in U_0 by

$$a_1 dz_1 + \cdots + a_n dz_n$$

and take the parametrized line $\ell = (t, 0, \ldots, 0)$. Then the tangencies of ω with ℓ are given by the zeros of $\omega_{|\ell} = a_1(t, 0, \ldots, 0)dt$. In principle, a_1 has degree $\leq d+1$, but the condition of contraction by the radial vector field gives us $a_0(t, 0, \ldots, 0) = -t a_1(t, 0, \ldots, 0)$, so $a_1(t, 0, \ldots, 0)$ has degree $\leq d$, and the number of points of tangencies is d. For more detailed discussion see [38, Proposition 1.2.1 p. 21].

2.3.5. Definition. The **singularities** of a projective one form ω are the zeros of the section

$$\omega : \mathcal{O}_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}(d+2).$$

Locally, if we write $\omega = a_1 dz_1 + \cdots + a_n dz_n$, the scheme of singularities of ω is defined by the ideal generated by $a_1, \ldots, a_n$. As a set, it consists of the common zeros of the a_i.

Exercise 7. As in the case of vector fields, show that a generic one form has isolated singularities. Using the sequence (2.9) and Proposition 1.6.10 show that the the number of singularities of a generic one form is the degree of the zero-cycle $c_n(\Omega_{\mathbb{P}^n}(d+2))$. Prove that this number is given by

$$(d+1)^n - (d+1)^{n-1} + \cdots + (-1)^{n-i}(d+1)^i + \cdots + (-1)^n.$$

2.3.6. Remark. In the case of integrable one forms, if $n \geq 3$ the set of singularities is not a finite set. In fact [32, p. 95] shows that there always exists a component of the singular set that has codimension two.

2.3.7. Definition. (Order of a singularity.)
Let p be a singularity of ω. Suppose that $p \in U_0$ and write

$$\omega = a_1 dz_1 + \cdots + a_n dz_n$$

for the local expression of ω in U_0. Then the **order of the singularity** p is

$$\nu_p(\omega) = \min\{\mathrm{ord}_p(a_i) \mid i = 1, \ldots, n\}.$$

It can be easily checked that this is independent of the choice of the open set U_j.

Some authors use *multiplicity* of the singularity instead of order.
In what follows we restrict ourselves to the case $n = 2$.

2.3.8. Definition. (Dicritical singularity) Let $\omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(d+2))$ and suppose that $p \in \mathbb{P}^2$ is a singularity of ω of order k. Let

$$\omega_p = a_k dx + b_k dy + h.o.t.$$

be a local expression of ω around $p = (0, 0)$. We say p is **dicritical of order** k if

$$a_k x + b_k y \equiv 0$$

(see [39, p. 47]). In the case $k = 1$, we say that p is a **radial singularity**.

Exercise 8. Prove that the dicriticity condition is equivalent to

$$\omega_p = f(x, y)(ydx - xdy) + h.o.t$$

for some homogeneous polynomial f of degree $k - 1$.

2.3.9. Definition. (Invariant hypersurface.)
Let $W \subset \mathbb{P}^2$ be an irreducible hypersurface defined by a homogeneous polynomial G of degree k, and $\mathcal{F}$ a foliation defined by a one form ω. We say that W **is invariant by** $\mathcal{F}$ if

$$\mathcal{T}_p W \subset \ker \omega_p$$

for all $p \in W \setminus (Sing(W) \cup Sing(\omega))$.

If W is reducible, we say that it is invariant by $\mathcal{F}$ if and only if each irreducible component of W is invariant by $\mathcal{F}$.

If G is irreducible, the above condition is equivalent to the existence of a two-form θ of degree d such that

$$dG \wedge \omega = G\theta.$$

See [32, p. 99].

2.3.10. Definition. (Points of tangency with a hypersurface.)

In the definition above, if W is not invariant, the two-form $dG \wedge \omega$ is not identically zero in W. The zeros of this form in $W \setminus (Sing(W))$ are the **tangencies of $\mathcal{F}$ with W**.

2.4. **Vector fields versus forms in $\mathbb{P}^2$.** In $\mathbb{P}^2$, a foliation can be defined by a vector field or by a one form.

In what follows we study one forms in $\mathbb{P}^2$ and their relation with vector fields.

Exercise 9. Prove that all one forms in $\mathbb{P}^2$ are automatically integrable *i.e.,* if

$$\omega = A_0 dZ_0 + A_1 dZ_1 + A_2 dZ_2$$

with homogeneous A_i of same degree, then

$$\iota_R(\omega) = 0 \implies \omega \wedge d\omega = 0.$$

Hence a foliation of degree d in $\mathbb{P}^2$ can be given by a vector field

$$\mathcal{X} \in H^0(\mathbb{P}^2, \mathcal{T}\mathbb{P}^2(d-1))$$

or by a one form

$$\omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(d+2)).$$

2.4.1. Remark. The reason behind this correspondence is the following. Recall that if $\mathcal{E}$ is a locally free sheaf of rank 2 we have a natural isomorphism

$$\mathcal{E} \simeq \mathcal{E}^{\vee} \otimes \overset{2}{\wedge}\, \mathcal{E}.$$

See [31, exercise 5.16 p. 127]. Taking $\mathcal{E} = \Omega_{\mathbb{P}^2}$ we obtain

$$\Omega_{\mathbb{P}^2} \simeq \mathcal{T}\mathbb{P}^2(-3).$$

Thus $\Omega_{\mathbb{P}^2}(d+2) \simeq \mathcal{T}\mathbb{P}^2(d-1)$.

If a foliation in $\mathbb{P}^2$ is given by a vector field

$$\mathcal{X} = F_0 \frac{\partial}{\partial Z_0} + F_1 \frac{\partial}{\partial Z_1} + F_2 \frac{\partial}{\partial Z_2}$$

and by a one form

$$\omega = A_0 dZ_0 + A_1 dZ_1 + A_2 dZ_2,$$

we may express the $F_i^{\prime}s$ in terms of the $A_i^{\prime}s$ and vice versa as follows.

Given $\mathcal{X}$ as above, the coefficients of ω are expressed by

$$\begin{cases} A_0 = Z_2 F_1 - Z_1 F_2, \\ A_1 = Z_0 F_2 - Z_2 F_0, \\ A_2 = Z_1 F_0 - Z_0 F_1. \end{cases}$$

Given ω, the coefficients of the vector field $\mathcal{X}$ can be obtained from

$$d\omega = (d+2)(F_0 dZ_1 \wedge dZ_2 + F_1 dZ_2 \wedge dZ_0 + F_2 dZ_0 \wedge dZ_1)$$

This is a consequence of the acyclicity of the Koszul complex associated to the regular sequence $\{Z_0, Z_1, Z_2\}$. See [32, §1.5].

Exercise 10. Prove that if ω and $\mathcal{X}$ define the same foliation in $\mathbb{P}^2$, then in U_0 we have

$$\omega_{U_0} = a_1 dz_1 + a_2 dz_2$$

and

$$\mathcal{X}_{U_0} = -a_2 \frac{\partial}{\partial z_1} + a_1 \frac{\partial}{\partial z_2}.$$

This corresponds to the intuitive idea that the vector field and the form defining a foliation are orthogonal to each other.

3. Foliations with degenerate singularities

If $\omega \in \mathbb{P}^N$, $(N = N_{1,2,d})$ defines a generic foliation of degree d in $\mathbb{P}^2$, its singularities are all nondegenerate (see [32]); in particular, they have all order one. In this Section we study foliations in $\mathbb{P}^2$ that have a degenerate singularity.

The first type of degeneration we will consider is to ask the order of the singularity to be some $k \geq 2$. The reader can easily check that, if ω_1, ω_2 are one forms that have order $\geq k$ at a point $p \in \mathbb{P}^2$, the same holds true for any linear combination $a_1\omega_1 + a_2\omega_2, a_i \in \mathbb{C}$. Thus, for fixed p, the condition is linear on the space of one forms.

This leads us to the correspondence $\mathbb{W}_k \subseteq \mathbb{P}^2 \times \mathbb{P}^N$ defined by the pairs (p, ω) such that the order of ω at p is $\geq k$. We define the locus $\mathbb{M}_k \subset \mathbb{P}^N$ of foliations with a singularity of order at least k as the image $p_2(\mathbb{W}_k) \subset \mathbb{P}^N$ by the second projection.

As expected from the previous discussion, $\mathbb{W}_k$ is actually a projective subbundle of $\mathbb{P}^2 \times \mathbb{P}^N$ over $\mathbb{P}^2$. More precisely, we have that $\mathbb{W}_k = \mathbb{P}(\mathcal{M}_k)$, the projectivization of a vector bundle $\mathcal{M}_k$ over $\mathbb{P}^2$. We are able to determine its characteristic classes, used to find the degree of $\mathbb{M}_k$.

In Section 3.2 we study the space $\mathbb{D}_k \subset \mathbb{M}_k$ of foliations that has a dicritical singularity of order k. Again, this is a closed condition, and we construct a parametrization of that space. We find a vector subbundle $\mathcal{D}_k \subset \mathcal{M}_k$ over $\mathbb{P}^2$, such that the image by the second projection of $\mathbb{P}(\mathcal{D}_k)$ is $\mathbb{D}_k$. We determine the characteristic classes of $\mathcal{D}_k$, and with this at hand we can compute the degree of $\mathbb{D}_k$.

Requiring a leaf of a foliation to be tangent to a line at a given point defines a hyperplane in $\mathbb{P}^N$. Thus, finding the degree of the loci $\mathbb{D}_k \subset \mathbb{M}_k$ can be rephrased loosely as calculating the number of foliations with a singularity of the chosen type and further tangent to the appropriate number of flags (point, line) in $\mathbb{P}^2$. It turns out that the degrees of $\mathbb{D}_k$ and $\mathbb{M}_k$ are expressed as explicit polynomials in k, d.

3.1. **Singularities of prescribed order.**

In order to simplify the notation we set

$$V := V_{1,2,d} = H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(d+2)) \quad \text{and} \quad N := N_{1,2,d} = \dim V - 1\,.$$

Fix $k \leq d+1$. In this section we describe a parameter space $\mathbb{M}_k \subset \mathbb{P}^N$ for the locus of foliations of given degree d that have some singularity of order $\geq k$. In fact we obtain a filtration of $\mathbb{P}^N$,

$$\mathbb{M}_{d+1} \subset \cdots \subset \mathbb{M}_3 \subset \mathbb{M}_2 \subset \mathbb{M}_1 = \mathbb{P}^N.$$

In Proposition (3.1.3, p. 57) we show that the codimension of $\mathbb{M}_k$ in $\mathbb{P}^N$ is

$$\mathrm{cod}\mathbb{M}_k = k(k+1) - 2$$

and

$$\deg(\mathbb{M}_k) = \int c_2(\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2))) \cap [\mathbb{P}^2]$$

where $\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2))$ is the $(k-1)-$jet bundle associated to $\Omega_{\mathbb{P}^2}(d+2)$, (cf. Appendix (A.1.4, p. 81)).

Recall (Definition 2.3.7, p. 53) that if $\omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(d+2))$, the order of a singularity $p \in \mathbb{P}^2$ is

$$\nu_p(\omega) := \min\{order_p(a), order_p(b)\},$$

where $\omega_p = adx + bdy$ is a local expression of ω in a neighborhood of p with $x(p) = y(p) = 0$.

Order one. Consider the map of fiber bundles over $\mathbb{P}^2$,

$$ev : \mathbb{P}^2 \times V \to \Omega_{\mathbb{P}^2}(d+2)$$

given by evaluation, $ev(p, \omega) = (p, \omega(p))$.

We claim that ev is surjective. In fact, it is sufficient to prove the surjectivity in the fibers. For this suppose $p = [0 : 0 : 1]$ and let $\lambda dx + \mu dy \in \Omega_p(d+2)$. Then

$$\omega := Z_2^d(Z_2\lambda dZ_0 + Z_2\mu dZ_1 - (Z_0\lambda + Z_1\mu)dZ_2) \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(d+2))$$

satisfies $\omega(p) = \lambda dx + \mu dy$.

3.1.1. Remark. The fact that $\Omega_{\mathbb{P}^2}(d+2)$ is generated by global sections can also be proven with cohomological tools, cf. [32, Lemme 2.3.6, p. 90].

Set $\mathcal{M} := \operatorname{Ker}(ev)$. Since ev is surjective, $\mathcal{M}$ is a subbundle of V of rank

$$\operatorname{rk}\mathcal{M} = \dim V - 2 = N - 1\,.$$

It fits into the following exact sequence

$$0 \to \mathcal{M} \longrightarrow \mathbb{P}^2 \times V \longrightarrow \Omega_{\mathbb{P}^2}(d+2) \to 0. \tag{3.1}$$

3.1.2. Definition. The **universal singular set** is the projective bundle associated to $\mathcal{M}$, *i.e.,* the incidence variety:

$$\mathbb{P}(\mathcal{M}) = \{(p, [\omega]) \mid p \text{ is singularity of } [\omega]\} \subset \mathbb{P}^2 \times \mathbb{P}^N.$$

Let us denote by p_1, q the projections of $\mathbb{P}(\mathcal{M})$ in the first and second factor respectively.

We have the diagram

$$\begin{tikzcd} & \mathbb{P}(\mathcal{M}) \arrow[dl, "p_1"'] \arrow[dr, "q"] & \\ \mathbb{P}^2 & & \mathbb{P}^N \end{tikzcd}$$

where q is surjective (all foliations have singularities) and generically finite (a generic foliation has isolated singularities) cf.[32].

We may compute the cardinality of a generic fiber of q (*i.e.,* $\deg(q)$) as follows. Observe that $q_*[\mathbb{P}(\mathcal{M})] = \deg(q)[\mathbb{P}^N]$. Write $H := c_1(\mathcal{O}_{\mathbb{P}^N}(1))$. Using properties of Chern classes and degree we have

$$\deg(q) = \int s_2(\mathcal{M}) \cap [\mathbb{P}^2].$$

The equality follows by Lemma (1.6.5, p. 45), recalling that $\mathrm{rk}(\mathcal{M}) = N-1$. In this way we retrieve the number of singularities of a general degree d foliation. Indeed, by sequence (3.1) we have $s_2(\mathcal{M}) = c_2(\Omega_{\mathbb{P}^2}(d+2))$. From the Euler sequence we find

$$c_2(\Omega_{\mathbb{P}^2}(d+2)) = d^2 + d + 1.$$

Order $k > 1$. Recall the exact sequence (see Appendix (A.1.4, p. 81)) for the jet bundles of $\Omega_{\mathbb{P}^2}(d+2)$,

$$(3.2) \quad 0 \to \mathrm{Sym}_n\, \Omega_{\mathbb{P}^2} \otimes \Omega_{\mathbb{P}^2}(d+2) \to \mathcal{P}^n(\Omega_{\mathbb{P}^2}(d+2)) \to \mathcal{P}^{n-1}(\Omega_{\mathbb{P}^2}(d+2)) \to 0$$

and the maps $ev_n : \mathbb{P}^2 \times V \to \mathcal{P}^n(\Omega_{\mathbb{P}^2}(d+2))$.

Exercise 11.

(1) Prove that ev_n is surjective for all $n \le d+1$.
(2) Prove that $ev_n(\mathrm{Ker}(ev_{n-1})) = \mathrm{Sym}_n\, \Omega_{\mathbb{P}^2} \otimes \Omega_{\mathbb{P}^2}(d+2)$.

3.1.3. Proposition. *For* $1 \le k \le d+1$, *denote by*

$$\mathbb{M}_k = \{[\omega] \in \mathbb{P}^N \mid [\omega] \text{ has a singularity of order at least } k\}.$$

Then we have

$$\mathrm{cod}_{\mathbb{P}^N}(\mathbb{M}_k) = k(k+1) - 2$$

and

$$\deg(\mathbb{M}_k) = \int_{\mathbb{P}^2} c_2(\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2)))\,.$$

Proof. Define

$$\mathcal{M}_k = \mathrm{Ker}\left(ev_{k-1} : \mathbb{P}^2 \times V \to \mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2))\right)\,.$$

In view of the previous exercise, we see that $\mathcal{M}_k$ is a vector subbundle of V of corank equal to $\mathrm{rk}\,\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2))$. By construction, the projective bundle associated to $\mathcal{M}_k$ is the incidence variety,

$$\mathbb{P}(\mathcal{M}_k) = \{(p,[\omega]) \in \mathbb{P}^2 \times \mathbb{P}^N \mid p \text{ is a singularity of } [\omega] \text{ and } \nu_p(\omega) \ge k\}.$$

Let $q : \mathbb{P}(\mathcal{M}_k) \to \mathbb{P}^N$ denote the projection in the second factor. We have $\mathbb{M}_k = q(\mathbb{P}(\mathcal{M}_k))$.

It is easy to check that q is generically injective (for $k > 1$) (cf. Lemma 3.1.4 below). It follows from Lemma 1.6.5, p. 45 that

$$\deg(\mathbb{M}_k) = \int s_2(\mathcal{M}_k) \cap [\mathbb{P}^2].$$

Since by definition of $\mathcal{M}_k$, $s_2(\mathcal{M}_k) = c_2(\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2)))$, the second assertion follows. The first assertion follows from the exact sequence (A.1), p. 81, and a simple inductive argument. □

3.1.4. Lemma. *For all* $2 \le k \le d+1$ *the projection* $q : \mathbb{P}(\mathcal{M}_k) \to \mathbb{P}^N$ *is generically injective.*

Proof. By Lemma 1.6.6, p. 45, we shall find a form ω of degree $d+1$ such that

(i): it has a unique singularity p of order k and
(ii): $d_{(p,[\omega])}q$ is injective.

Suppose that $k>2$. We claim that the following form fulfills (i) and (ii):

$$\omega=(Z_2^{d-(k-1)}Z_0^{k-1}+Z_2^{d-(k-1)}Z_1^{k-1}+Z_0^d+Z_1^d)(-Z_1dZ_0+Z_0dZ_1).$$

Indeed, write $F:=Z_2^{d-(k-1)}Z_0^{k-1}+Z_2^{d-(k-1)}Z_1^{k-1}+Z_0^d+Z_1^d$. In the chart U_2 we have

$$\omega=(x^{k-1}+y^{k-1}+x^d+y^d)(-ydx+xdy):=f\cdot(-ydx+xdy):=adx+bdy.$$

Is clear that $(0,0)$ is a singularity of order k. Suppose that $(\alpha,\beta)\in U_2=\mathbb{C}^2$ is another singularity. Then we have

$$f(\alpha,\beta)=\alpha^{k-1}+\beta^{k-1}+\alpha^d+\beta^d=0.$$

On the other hand, if the first jet of ω at that point is zero we have

$$\begin{cases}\dfrac{\partial a}{\partial x}(\alpha,\beta) &= -\beta\dfrac{\partial f}{\partial x}(\alpha,\beta)=-\beta\alpha^{k-2}(k-1+d\alpha^{d-k+1})=0\\ \dfrac{\partial a}{\partial y}(\alpha,\beta) &= -\beta\dfrac{\partial f}{\partial y}(\alpha,\beta)=-\beta^{k-1}(k-1+d\beta^{d-k+1})=0\\ \dfrac{\partial b}{\partial x}(\alpha,\beta) &= \alpha\dfrac{\partial f}{\partial x}(\alpha,\beta)=\alpha^{k-1}(k-1+d\alpha^{d-k+1})=0\\ \dfrac{\partial b}{\partial y}(\alpha,\beta) &= \alpha\dfrac{\partial f}{\partial y}(\alpha,\beta)=\alpha\beta^{k-2}(k-1+d\beta^{d-k+1})=0.\end{cases}$$

Suppose that $\alpha\neq 0$. Then $\frac{\partial f}{\partial x}(\alpha,\beta)=0$. If (α,β) is a singularity of order greater than two, then

$$\frac{\partial^2 b}{\partial^2 x}(\alpha,\beta)=\alpha\frac{\partial^2 f}{\partial^2 x}(\alpha,\beta)=\alpha^{k-2}((k-1)(k-2)+d(d-1)\alpha^{d-k+1})=0$$

i.e.,

$$((k-1)(k-2)+d(d-1)\alpha^{d-k+1})=0.$$

This, together with $d\alpha^{d-k+1}=-(k-1)$ implies $k-2-(d-1)=k-1-d=0$ *i.e.*, $k=d+1$. It is easy to see that if $k=d+1$ the unique singularity of order ≥ 1 is $(0,0)$. Therefore $\alpha=0$, and similarly $\beta=0$.

On the other charts, for example in U_0 the expression of the form is

$$(z^{d-k+1}(1+y^{k-1})+1+y^d)dy.$$

It is easy to see that the singularities are of order less than two.

It remains to show that $d_{(p,[\omega])}q$ is injective. For this, we consider a vector $((p_1,p_2),\theta))\in\mathcal{T}_{(p,[\omega])}\mathbb{P}(\mathcal{M}_k)$, and we have to prove that $\theta=0$ implies $p_1=p_2=0$.

The vector above is the tangent vector to a curve $([\varepsilon p_1:\varepsilon p_2:1],\omega+\varepsilon\theta)$ in $\mathbb{P}(\mathcal{M}_k)$ if and only if the point is a singularity of order $\geq k$ (working in $\mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle$).

Suppose that $J_i(\omega+\varepsilon\theta)(\varepsilon p_1,\varepsilon p_2)=0$ for all $i\leq k-1$ (here $J_i(\omega)$ stands for the part of order i of ω). It is easy to see that in this case, if $\theta=0$, $J_{k-1}(\omega+\varepsilon\theta)(\varepsilon p_1,\varepsilon p_2)=0$ implies

$$\begin{cases}-\varepsilon p_2\frac{\partial^{k-1}f}{\partial^{k-1}x}(\varepsilon p_1,\varepsilon p_2)=0\\ -\varepsilon p_1\frac{\partial^{k-1}f}{\partial^{k-1}x}(\varepsilon p_1,\varepsilon p_2)=0.\end{cases}$$

But $\frac{\partial^{k-1}f}{\partial^{k-1}x} = ((k-1)! + d(d-1)\cdots(d-k+2)x^{d-k+1})$. Hence the above equations imply

$$\begin{cases} \varepsilon p_2(k-1)! = 0 \\ \varepsilon p_1(k-1)! = 0 \end{cases}$$

i.e., $p_1 = p_2 = 0$.

For $k = 2$, the form

$$\omega := (Z_0Z_2^{d-1} + Z_1Z_2^{d-1} + Z_0^d + (-Z_1)^d)(-Z_1dZ_0 + Z_0dZ_1)$$

has the required properties.

□

Employing the proposition above, we may now derive an explicit formula for the degree of $\mathbb{M}_k \subset \mathbb{P}^N$. We use SCHUBERT, [34] to do the computations, see the script in [16]. We find

3.1.5. Corollary. *The degree of* $\mathbb{M}_k$ *is*

$$\frac{1}{2}k(k+1)\left[(k^2+k-1)\left(d^2-(2k-3)d\right)+\frac{1}{4}(4k^4-8k^3-7k^2+21k-6)\right].$$

□

3.2. **Dicritical singularities.** Recall that if $\omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(d+2))$ and p is a singularity of ω, we say that p is dicritical if the local expression of ω around p is

$$\omega_p = a_kdx + b_kdy + h.o.t$$

with $a_kx + b_ky = 0$ (Definition 2.3.8, p. 53). To have a dicritical singularity will be shown to be a closed condition in $\mathbb{P}^N$. This will be rephrased shortly in a coordinate-free manner.

In this section we describe the locus $\mathbb{D}_k$ of forms of given degree that have a dicritical singularity of order k.

In Proposition (3.2.3, p. 60) we obtain that the codimension of $\mathbb{D}_k$ is

$$k(k+2)$$

and the degree of $\mathbb{D}_k$ is given by the coefficient of the degree two part of

$$c(\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2)))c(\mathrm{Sym}_{k+1}\,\Omega_{\mathbb{P}^2}\otimes\mathcal{O}_{\mathbb{P}^2}(d+2)).$$

3.2.1. Remark. Next we explain an invariant way to express the condition that a singularity is dicritical.

Suppose that $\mathcal{E}$ is a vector bundle of rank 2. Then for all $k \geq 1$ we have the following exact sequence (*e.g.*, see [13, Appendix 2 A2.6.1.]):

$$0 \to \bigwedge^2 \mathcal{E} \otimes \mathrm{Sym}_{k-1}\,\mathcal{E} \to \mathrm{Sym}_k\,\mathcal{E}\otimes\mathcal{E} \xrightarrow{P_k} \mathrm{Sym}_{k+1}\,\mathcal{E} \to 0,$$

where the first map is given by

$$(a \wedge b \otimes c) \mapsto (ac \otimes b) - (bc \otimes a)$$

and the second by

$$a \otimes b \mapsto ab.$$

Say x, y form a local basis for $\mathcal{E}$. Then for $a_k, b_k \in \mathrm{Sym}_k\,\mathcal{E}$, we have that $a_kx + b_ky = 0$ in $\mathrm{Sym}_{k+1}\,\mathcal{E}$ if and only if there is some $c \in \mathrm{Sym}_{k-1}\,\mathcal{E}$ such that $a_k \otimes x + b_k \otimes y$ is equal to the image of $x \wedge y \otimes c$, to wit, $xc \otimes y - yc \otimes x$. cf. Exercise (8, p. 53).

3.2.2. Lemma. *For all* $1 \leq k \leq d$ *there exists a subbundle* $\mathcal{D}_k$ *of the trivial bundle* $\mathbb{P}^2 \times V$ *such that*

$$\mathbb{P}(\mathcal{D}_k) = \{(p, [\omega]) \mid p \text{ is a dicritical singularity of } [\omega] \text{ with } \nu_p(\omega) \geq k\} \subset \mathbb{P}^2 \times \mathbb{P}^N.$$

Proof. From the previous section we have the following diagram,

$$\begin{tikzcd}
 & & \operatorname{Sym}_k \Omega_{\mathbb{P}^2} \otimes \Omega_{\mathbb{P}^2}(d+2) \arrow[d, tail] \\
 & & \mathcal{P}^k(\Omega_{\mathbb{P}^2}(d+2)) \arrow[d, two heads] \\
\mathcal{M}_k \arrow[r, tail] \arrow[uurr, bend left, two heads, "J_k"] & V \arrow[r, two heads] \arrow[ur, bend left, two heads, "ev_k"] & \mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2)).
\end{tikzcd}$$

where the map J_k is surjective in view of Exercise 11.

We obtain the surjective map

$$\begin{tikzcd}
\mathcal{M}_k \arrow[r, "J_k"] \arrow[rr, bend right, two heads, "T_k"'] & \operatorname{Sym}_k \Omega_{\mathbb{P}^2} \otimes \Omega_{\mathbb{P}^2}(d+2) \arrow[r, "P_k"] & \operatorname{Sym}_{k+1} \Omega_{\mathbb{P}^2}(d+2).
\end{tikzcd}$$

Explicitly, on the fiber over $p \in \mathbb{P}^2$ the map is as follows:

$$T_k(p, \omega) = (p, a_k x + b_k y)$$

where

$$\omega_p = a_k dx + b_k dy + h.o.t.$$

is the local expression of ω in a neighborhood of p with $x(p) = y(p) = 0$. Set

$$\mathcal{D}_k := \ker\left(\mathcal{M}_k \xrightarrow{T_k} \operatorname{Sym}_{k+1} \Omega_{\mathbb{P}^2}(d+2)\right) \tag{3.3}$$

Thus $\mathcal{D}_k$ is a vector bundle of rank $= \operatorname{rk}(\mathcal{M}_k) - (k+2)$. Recalling (3.2.1, p. 59), we see that the projective bundle associated to $\mathcal{D}_k$ is the incidence variety,

$$\mathbb{P}(\mathcal{D}_k) = \{(p, [\omega]) \in \mathbb{P}^2 \times \mathbb{P}^N \mid p \text{ is a dicritical singularity of } [\omega] \text{ with } \nu_p(\omega) \geq k\}.$$

$\square$

For $1 \leq k \leq d+1$, denote by

$$\mathbb{D}_k = \{[\omega] \in \mathbb{P}^N \mid [\omega] \text{ has a dicritical singularity of order at least } k\}.$$

3.2.3. Proposition. *The degree of* $\mathbb{D}_k$ *is the coefficient of the degree two part of*

$$c(\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2)))c(\operatorname{Sym}_{k+1} \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)).$$

The codimension of $\mathbb{D}_k$ *is* $k(k+2)$.

Proof. From the above construction we have the maps

$$\begin{tikzcd}
 & \mathbb{P}(\mathcal{D}_k) \subset \mathbb{P}^2 \times \mathbb{P}^N \arrow[dl, "p_1"'] \arrow[dr, "q"] & \\
\mathbb{P}^2 & & \mathbb{D}_k \subset \mathbb{P}^N.
\end{tikzcd}$$

If q is generically injective the degree of $\mathbb{D}_k$ is computed as

$$\int s_2(\mathcal{D}_k) \cap [\mathbb{P}^2]$$

(see Lemma (1.6.5, p. 45)). By construction $\mathcal{D}_k$ fits into the following exact sequence,

$$0 \to \mathcal{D}_k \to \mathcal{M}_k \to \operatorname{Sym}_{k+1} \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2) \to 0.$$

Hence

$$s(\mathcal{D}_k) = s(\mathcal{M}_k) c(\operatorname{Sym}_{k+1} \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)),$$

and from the proof of Proposition 3.1.3 we have $s(\mathcal{M}_k) = c(\mathcal{P}^{k-1}(\Omega_{\mathbb{P}^2}(d+2)))$.

On the other hand, $\operatorname{rk}(\mathcal{D}_k) = \operatorname{rk}(\mathcal{M}_k) - (k+2)$. Thus

$$\operatorname{cod}_{\mathbb{P}^N} \mathbb{D}_k = \operatorname{cod} \mathbb{M}_k + (k+2) = k(k+1) - 2 + (k+2) = k(k+2).$$

It remains to prove that q is generically injective, but this follows from the generic injectivity of the projection $\mathbb{P}(\mathcal{M}_k) \to \mathbb{M}_k$. Indeed, let $U \subset \mathbb{M}_k$ denote the open set where the fiber of $\mathbb{P}(\mathcal{M}_k) \to \mathbb{M}_k$ consists of just one reduced point. Now observe that the examples constructed in Lemma (3.1.4, p. 57) are in $U \cap \mathbb{D}_k$. Hence $U \cap \mathbb{D}_k$ is a non empty open set over which the fibers of $q : \mathbb{P}(\mathcal{D}_k) \to \mathbb{D}_k$ consist of one reduced point. □

We may now compute an explicit formula for the degree of $\mathbb{D}_k \subset \mathbb{P}^N$ using Schubert, [34]. See the script in [16]. We find

3.2.4. Corollary. *The degree of $\mathbb{D}_k$ is given by*

$$(k+1)^2 \left[\frac{1}{2}(k^4 + k^2 - 2k + 2) - (k^3 + k^2 + k - 1)d + \frac{1}{2}(k^2 + 2k + 2)d^2 \right].$$

□

3.2.5. Remarks. **(i)** We have by construction the following diagram:

$$\begin{tikzcd}
 & & \operatorname{Sym}_{k-1} \Omega_{\mathbb{P}^2} \otimes \overset{2}{\wedge} \Omega_{\mathbb{P}^2}(d+2) \arrow[d] \\
\mathcal{D}_k \arrow[r] \arrow[urr, bend left, "d_k"] \arrow[drr, "0"'] & \mathcal{M}_k \arrow[r, "J_k"] \arrow[dr, "T_k"] & \operatorname{Sym}_k \Omega_{\mathbb{P}^2} \otimes \Omega_{\mathbb{P}^2}(d+2) \arrow[d, "P_k"] \\
 & & \operatorname{Sym}_{k+1} \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)
\end{tikzcd}$$

By definition of $\mathcal{D}_k$ we get a map

$$d_k : \mathcal{D}_k \to \operatorname{Sym}_{k-1} \Omega_{\mathbb{P}^2} \otimes \overset{2}{\wedge} \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$$

given in the fibers by $d_k(p, \omega) = f(x,y) dx \wedge dy$ where f is a polynomial of degree $k - 1$.

(ii) In the case $k = 1$ we have

$$\omega = \lambda(y dx - x dy) + h.o.t.$$

with $\lambda \in \mathbb{C}$, *i.e.,* a radial singularity (see Definition (2.3.8, p. 53)). Thus Corollary 3.2.4 and Proposition 3.2.3 give formulas for the codimension and degree of the space of foliations with a radial singularity:

$$\begin{cases} \operatorname{cod}_{\mathbb{P}^N} \mathbb{D}_1 = 3 \\ \deg \mathbb{D}_1 = 10d^2 - 8d + 4. \end{cases}$$

(iii) In the case $k = d+1$ the map

$$J_{d+1} : \mathcal{M}_{d+1} \to \operatorname{Sym}_{d+1} \Omega^1_{\mathbb{P}^2} \otimes \Omega_{\mathbb{P}^2}(d+2)$$

is no longer surjective: its image is $\operatorname{Sym}_d \Omega_{\mathbb{P}^2} \otimes \overset{2}{\wedge} \Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$. Indeed, suppose that ω is a form of degree $d+1$ which has p as singularity of order $d+1$. Then a local expression of ω is

$$\omega_p = a_{d+1} dx + b_{d+1} dy \,,$$

but this form defines a projective form of degree $d+1$ in $\mathbb{P}^2$ if and only if

$$a_{d+1} x + b_{d+1} y = 0$$

i.e., if p is a dicritical singularity. Therefore we can write

$$\omega_p = f(x,y)(ydx - xdy)$$

for some homogeneous polynomial f of degree d, *i.e.,* $\omega_p \in \operatorname{Sym}_d \Omega^1_p \otimes \overset{2}{\wedge} \Omega^1_p$. Hence

$$T_{d+1} : \mathcal{M}_{d+1} \xrightarrow{J_{d+1}} \operatorname{Sym}_{d+1} \Omega_{\mathbb{P}^2} \otimes \Omega_{\mathbb{P}^2}(d+2) \xrightarrow{P_{d+1}} \operatorname{Sym}_{d+2} \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$$

is the zero map. This shows that $\mathcal{M}_{d+1} = \mathcal{D}_{d+1}$, *i.e.,* for a foliation of degree d a singularity of order $d+1$ is automatically dicritical.

4. Foliations with invariant algebraic subvarieties

Jouanoulou shows in [32] that the set of foliations that do not have any invariant algebraic curve is dense in the ordinary topology in the variety that parametrizes foliations of degree $d \geq 2$ in $\mathbb{P}^2$. In [37] Lins-Neto proves that this set contains an open and dense subset. Finally, in [9] Coutinho and Pereira show that in a smooth complex projective variety of dimension grater or equal to two, a generic foliation of dimension one and sufficiently ample cotangent bundle (in the case of $\mathbb{P}^n$ this means degree big enough) has no invariant algebraic subvarieties of positive dimension. In fact the result of Jouanoulou (Coutinho and Pereira respectively) states that the set U_k of dimension one foliations in $\mathbb{P}^2$ ($\mathbb{P}^n$ respectively) without algebraic solution of degree k is an open set in the Zariski topology. We then ask for the complementary of the open set U_k. *i.e.,* we want to study the subset of the space of foliations of dimension one and degree $d \geq 2$ in $\mathbb{P}^n$ that has an algebraic solution of fixed degree and dimension.

To be more precise, we fix some type of positive dimensional subvarieties in $\mathbb{P}^n$. By this we mean an irreducible family of subvarieties, say hypersurfaces in $\mathbb{P}^n$ of given degree. It turns out that the subset of the space of one dimensional foliations of sufficiently high fixed degree that do have an invariant subvariety of a given "type" is irreducible. We would like to determine its codimension and degree. For this, we try to find an adequate description of these subvarieties in the spirit of the previous Section, namely, as the birrational image of projective bundles associated to vector bundles over the variety that parameterizes the desired invariant subvarieties, e.g. Grassmannians for linear spaces, "complete conics" for conics.

Heuristically, requiring a fixed subvariety to be invariant by a foliation amounts to imposing linear conditions on the coefficients of a vector field defining the foliation. It is reasonable to expect that the number of independent conditions remains fixed as the subvariety varies in a suitable open subset of its parameter space. This is clearly the case if the type of subvariety we wish to be invariant consists of a single orbit under the group of automorphism of $\mathbb{P}^n$. For instance, imposing a linear subspace of fixed dimension does produce a nice projective bundle over the corresponding Grassmannian. However, already for hypersurfaces of any degree ≥ 2, the number of independent conditions jumps in the presence of singularities of the variety.

This question, in full generality, seems to be complicated. In this Section we solve the problem of describing foliations that have invariant subsets of degree 1 in $\mathbb{P}^n$ and of degree 2 in $\mathbb{P}^2$.

In the first Subsection we find a parameter space for foliations with linear invariant subset of any fixed dimension in $\mathbb{P}^n$.

In Subsection two, we deal with the problem of foliations in $\mathbb{P}^2$ with an invariant conic. With the same techniques it is possible to describe the space of foliations with invariant quadrics in $\mathbb{P}^n$. We will give the formulas for the degree and codimension of the space of foliations with invariant conic (respectively, quadric) in $\mathbb{P}^3$ in the last part of this Subsection.

4.1. **Foliations with invariant linear subspaces.** Fix $1 \leq r < n$. Set for short

$$(4.1) \qquad \begin{cases} N := N_{1,n,d}, \ V := V_{1,n,d} \text{ cf. } (2.7, 2.4, \text{p. } 48) \\ \mathbb{G} := \mathbb{G}(r,n), \ \text{the Grassmannian of } r\text{-dimensional subspaces of } \mathbb{P}^n. \end{cases}$$

Assume $d \geq 2$; for the cases $d = 0, 1$ see Exercise (13, p. 67). We define

$$\widehat{\mathbb{W}} := \{(W, [\mathcal{X}]) \in \mathbb{G} \times \mathbb{P}^N \mid W \text{ is invariant by } \mathcal{X}\}.$$

The goal of this subsection is to prove the following

4.1.1. Proposition. *Notation as above, there exists a vector subbundle*

$$\mathcal{E} \subset \mathbb{G} \times V$$

such that

(i) $\mathbb{P}(\mathcal{E}) = \widehat{\mathbb{W}} \subset \mathbb{G} \times \mathbb{P}^N$ *;*
(ii) *if we set* $\mathbb{W} := q(\mathbb{P}(\mathcal{E}))$*, where* $q : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N$ *is the projection, then the codimension of* $\mathbb{W}$ *in* $\mathbb{P}^N$ *is*

$$\mathrm{cod}_{\mathbb{P}^N} \mathbb{W} = (n-r)(\binom{r+d}{d} - (r+1))\,;$$

(iii) *and the degree of* $\mathbb{W}$ *is given by the top-dimensional Chern class,*

$$\deg \mathbb{W} = \int c_g(\mathcal{Q} \otimes \mathrm{Sym}_d(\mathcal{S}^\vee)) \cap [\mathbb{G}],$$

where $g := \dim \mathbb{G}$.

The image $\mathbb{W}$ of $\widehat{\mathbb{W}}$ in $\mathbb{P}^N$ via projection is the set of dimension one degree d foliations in $\mathbb{P}^n$ that have an invariant r-plane.

Proof. Consider the tautological sequence over $\mathbb{G}$ (A.7, p. 84) and take its dual sequence

$$0 \to \mathcal{Q}^\vee \to \mathbb{G} \times S_1 \to \mathcal{S}^\vee \to 0. \tag{4.2}$$

The fiber of $\mathcal{Q}^\vee$ over $W \in \mathbb{G}$ is the space of equations that define W. The map $\mathbb{G} \times S_1 \twoheadrightarrow \mathcal{S}^\vee$ induces a surjective map

$$\mathbb{G} \times S_d \twoheadrightarrow \mathrm{Sym}_d(\mathcal{S}^\vee).$$

On the fiber over $W \in \mathbb{G}$ it is the map of restriction of degree d polynomials to W. On the other hand, from the tautological sequence,

$$0 \to \mathcal{S} \to \mathbb{G} \times S_1^\vee \to \mathcal{Q} \to 0,$$

setting $\Lambda = \mathcal{S}_W$, using (2.1), p. 48, we can interpret the surjective map

$$S_1^\vee \twoheadrightarrow \mathcal{Q}_W$$

as the quotient of $\mathcal{T}\mathbb{C}^{n+1}$ by $\mathcal{T}\Lambda$. Tensoring these two maps we obtain a surjective map of vector bundles over $\mathbb{G}$, $\varphi : S_d \otimes S_1^\vee \twoheadrightarrow \mathrm{Sym}_d(\mathcal{S}^\vee) \otimes \mathcal{Q}$, which in the fiber over W is given by $\varphi_W(F \otimes X) = F_{|W} \otimes \overline{X}$. It is easy to see that $\varphi(S_{d-1} \cdot R) \equiv 0$ (the radial field restricted to Λ is the radial field in Λ). Hence we obtain a surjective map of vector bundles, $\psi : \mathbb{G} \times V \twoheadrightarrow \mathrm{Sym}_d(\mathcal{S}^\vee) \otimes \mathcal{Q}$. It's not hard to check that the following are equivalent:

- $\psi_W(\mathcal{X}) = 0$
- $X_{|\Lambda} \in T\Lambda$
- Λ is invariant by X
- $W = \mathbb{P}(\Lambda)$ is invariant by $\mathcal{X}$.

Therefore $\mathcal{E} := \mathrm{Ker}(\psi)$ is a subbundle of V with $\mathbb{P}(\mathcal{E}) = \widehat{\mathbb{W}}$. This proves (i). We also get the rank of $\mathcal{E}$ is $\dim V - (n-r)\binom{d+r}{d}$. Assertions (ii) and (iii) will be dealt with below. □

4.1.2. The degree of $\mathbb{W}$. In order to compute the degree of $\mathbb{W}$ it remains to prove that q is generically injective, and then apply Lemma (1.6.5, p. 45).

4.1.3. Lemma. *Notation as in* Prop. (4.1.1, p. 63)*, for $d > 1$, the projection $q : \widehat{\mathbb{W}} \to \mathbb{P}^N$ is generically injective.*

Proof. Recall that $\mathbb{W} = q(\widehat{\mathbb{W}})$. It's sufficient to prove that there exists an open set $U_1 \subset \mathbb{W}$ such that for each point $y \in U_1$, $q^{-1}(y)$ consists of just one point. Indeed, in this case we deduce that $\dim \widehat{\mathbb{W}} = \dim \mathbb{W}$. Since $\widehat{\mathbb{W}}$ (resp. $\mathbb{W}$) are smooth (resp. generically smooth) varieties we have that

$$dq : \mathcal{T}\widehat{\mathbb{W}} \longrightarrow \mathcal{T}\mathbb{W}$$

is generically of maximal rank, *i.e.,* there exists an open set $U_2 \subset \mathbb{W}$ such that if $y \in U_2$, and $x \in q^{-1}(y)$, then $d_x q$ is surjective, equivalently $d_x q$ is injective. Summarizing we find an open set $U := U_1 \cap U_2$ such that if $y \in U$ then $q^{-1}(y) = \{x\}$, and x is a reduced point in the fiber. It follows that q is generically injective cf. (1.6.6, p. 45).

Let's prove the existence of U_1 above. For $W \in \mathbb{G}$, set $\mathbb{W}_W \subset \mathbb{W}$ the set of foliations which leave W invariant. For each $W' \neq W$ define $\mathbb{W}_{WW'} := \mathbb{W}_W \cap \mathbb{W}_{W'}$ and

$$\mathbb{W}_2 := \{\mathcal{X} \in \mathbb{W} \mid \mathcal{X} \in \mathbb{W}_{WW'} \text{ for some } W \neq W'\}.$$

We claim that $\dim \mathbb{W}_2 < \dim \mathbb{W}$. Indeed, define

$$G_2 := \{(W, W') \in \mathbb{G} \times \mathbb{G} \mid W \neq W'\},$$

$$\widehat{\mathbb{W}}_2 = \{(\mathcal{X}, (W, W')) \mid \mathcal{X} \in \mathbb{W}_{WW'}; (W, W') \in G_2\}.$$

For each $1 \le s \le \min\{r, n-r\}$ set

$$G_{2s} := \{(W, W') \in \mathbb{G} \times \mathbb{G} \mid \mathrm{cod}(W \cap W') = n - r + s\},$$

$$\widehat{\mathbb{W}}_{2s} = \{(\mathcal{X}, (W, W')) \mid \mathcal{X} \in \mathbb{W}_{WW'}; (W, W') \in G_{2s}\}.$$

Let $p : \widehat{\mathbb{W}}_{2s} \to \mathbb{W} \subset \mathbb{P}^N$ denote the projection. Then we have, for each s

$$\begin{tikzcd} & \widehat{\mathbb{W}}_{2s} \arrow[dl] \arrow[dr, "p"] & \\ G_{2s} & & \mathbb{W} \end{tikzcd}$$

It is easy to see that $\dim G_{2s} = r(n-r) + s(r-s) = m + s(r-s)$. Furthermore each $\widehat{\mathbb{W}}_{2s} \to G_{2s}$ is a fibration with fiber dimension equal to $\dim \mathbb{W}_{WW'}$. We claim that

$$\dim \mathbb{W}_{WW'} \le \dim \mathbb{W}_W - s\binom{r+d}{d}.$$

In fact, suppose that $W = \mathcal{Z}(Z_0, \dots, Z_{n-r-1})$. Then W is invariant by a field of degree d,

$$\mathcal{X} = F_0 \frac{\partial}{\partial Z_0} + \cdots + F_n \frac{\partial}{\partial Z_n}$$

if and only if

$$F_0, \dots, F_{n-r-1} \in \langle Z_0, \dots, Z_{n-r-1} \rangle. \tag{4.3}$$

Take W' such that $\mathrm{cod}(W \cap W') = n - r + s$. Acting with the stabilizer of W in PGL_{n+1} we can suppose that $W = \mathcal{Z}(Z_{i_1}, \dots, Z_{i_{n-r}})$, with $W' \cap W = \mathcal{Z}(Z_{i_1}, \dots, Z_{i_s})$ where $i_1, \dots, i_s \notin \{0, \dots, n-r-1\}$. Now the condition of W' to be invariant by $\mathcal{X}$ implies that, for each $j = 1, \dots, s$ we have

$$F_{i_j} \in \langle Z_{i_1}, \dots, Z_{i_{n-r}} \rangle.$$

These conditions are independent of the others in (4.3). On the other hand it is easy to count the new conditions imposed by $F_{i_j} \in \langle Z_{i_1}, \dots, Z_{i_{n-r}} \rangle$: this number is $\binom{r+d}{d}$. So we have that $\mathrm{cod}_{\mathbb{W}_W} \mathbb{W}_{WW'} \ge s\binom{r+d}{d}$. This proves the claim.

Next, let ϵ denote the dimension of the generic fiber of q and ϵ_2 the dimension of the generic fiber of p. It is clear that $\epsilon_2 \ge \epsilon$. Hence

$$\begin{aligned} \dim \mathbb{W}_{2s} &= \dim \widehat{\mathbb{W}}_{2,s} - \epsilon_2 \\ &\le m + s(r-s) + \dim \mathbb{W}_W - s\tbinom{r+d}{d} - \epsilon \\ &= \dim \mathbb{W} + s(r-s) - s\tbinom{r+d}{d}. \end{aligned}$$

Therefore, in order to prove that $\dim \mathbb{W}_2 < \dim \mathbb{W}$ it is enough to prove that

$$s(r-s) - s\binom{r+d}{d} < 0. \tag{4.4}$$

As $s \ge 1$ we have $(r-s) \le (r-1) < (r+1)$. On the other hand, we have

$$\begin{aligned} (r+d)\cdots(r+2)(r+1) &\ge (d+1)\cdots 3\,(r+1) \\ &\ge d!\,(r+1) > d!\,(r-s). \end{aligned}$$

This proves (4.4). Now, define $U_1 := \mathbb{W} \setminus \mathbb{W}_2$. Clearly U_1 is an open dense subset of $\mathbb{W}$ such that if $\mathcal{X} \in U_1$ then $\#q^{-1}(\mathcal{X}) = 1$. $\square$

We can now complete the proof of (ii) and (iii) of the Prop. (4.1.1, p. 63). By construction of $\mathcal{E}$ (cf. p. 63) we have the following diagram:

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}) = \widehat{\mathbb{W}} \arrow[dl, "p_1"'] \arrow[dr, "q"] & \\ \mathbb{G} & & \mathbb{W} \end{tikzcd}$$

and we have proved that q is generically injective. Therefore

$$\operatorname{cod}_{\mathbb{P}^N} \mathbb{W} = N - \dim(\mathbb{P}(\mathcal{E})) = N - (g + \operatorname{rk}(\mathcal{E}) - 1) =$$

$$N - \left(g + (N - (n-r)\binom{r+d}{d})\right) = (n-r)\binom{r+d}{d} - g$$

where $g = \dim \mathbb{G} = (n-r)(r+1)$.

By Lemma (1.6.5, p. 45) the degree of $\mathbb{W}$ is equal to

$$\int s_g(\mathcal{E}) \cap [\mathbb{G}].$$

From the exact sequence that defines $\mathcal{E}$,

$$0 \to \mathcal{E} \to V \to \mathcal{Q} \otimes \operatorname{Sym}_d(\mathcal{S}^\vee) \to 0$$

we get

$$\int s_g(\mathcal{E}) \cap [\mathbb{G}] = \int c_g(\mathcal{Q} \otimes \operatorname{Sym}_d(\mathcal{S}^\vee)) \cap [\mathbb{G}] \,.$$

$\square$

4.1.4. Examples Next we give explicitly some codimensions and degrees. We use a script for SCHUBERT, [34] (see [16]) for the computations.

($\mathbb{P}^2$, $r = 1$)

$$\operatorname{cod}_{\mathbb{P}^N} \mathbb{W} = d - 1$$
$$\deg \mathbb{W} = \frac{1}{8} d(d+1)(d+2)(d+3)$$

($\mathbb{P}^3$, $r = 1$)

$$\operatorname{cod}_{\mathbb{P}^N} \mathbb{W} = 2(d-1)$$
$$\deg \mathbb{W} = \frac{1}{36} d(d+2)(d+1)(3d^5 + 9d^4 + 11d^3 + 9d^2 - 11d + 15)$$

($\mathbb{P}^3$, $r = 2$)

$$\operatorname{cod}_{\mathbb{P}^N} \mathbb{W} = \frac{1}{2}(d+4)(d-1)$$
$$\deg \mathbb{W} = \frac{1}{6^4} d(d+3)(d+2)(d+1)(d^2 + 6d + 11)(d^3 + 6d^2 + 11d - 6)$$

($\mathbb{P}^4$, $r = 1$)

$$\operatorname{cod}_{\mathbb{P}^N} \mathbb{W} = 3(d-1)$$
$$\deg \mathbb{W} = \tfrac{1}{2^{10}.3^2} d(d+2)(d+1)\left[729d^9 + 2187d^8 + 3402d^7 + 3750d^6 - 279d^5 + 651d^4 - 2668d^3 + 9732d^2 - 8864d + 6720\right].$$

Exercise 12. In the case of a hyperplane in $\mathbb{P}^n$, $\mathrm{rk}(\mathcal{Q}) = 1$,($\mathcal{Q} = \mathcal{O}_{\check{\mathbb{P}}^n}(1)$) we have the following exact sequence:

$$0 \to S_{d-1} \otimes \mathcal{Q}^\vee \to S_d \to \mathrm{Sym}_d(\mathcal{S}^\vee) \to 0.$$

Twisting by $\mathcal{Q}$ we obtain:

$$0 \to S_{d-1} \to S_d \otimes \mathcal{Q} \to \mathrm{Sym}_d(\mathcal{S}^\vee) \otimes \mathcal{Q} \to 0.$$

Use this to prove that $c_n(\mathrm{Sym}_d(\mathcal{S}^\vee) \otimes \mathcal{Q}) = c_n(S_d \otimes \mathcal{Q}) = \binom{\binom{d+n}{n}}{n}$.

Exercise 13. In this exercise we review the case of foliations of degree 0.

Prove that a degree 0 foliation given by a field

$$\mathcal{X} = \lambda_0 \frac{\partial}{\partial Z_0} + \lambda_1 \frac{\partial}{\partial Z_1} + \cdots + \lambda_n \frac{\partial}{\partial Z_n}$$

with $\lambda_i \in \mathbb{C}$, is radial with center $p := [\lambda_0 : \lambda_1 : \cdots : \lambda_n]$.

It follows that any line trough p is invariant. Therefore in this case the map q is infinity to one.

In the case of foliations of degree 1 the well known correspondence between the set of such foliations, and the space of $(n+1)\times(n+1)$ matrices of trace zero (see [32, p. 9]), shows that a generic foliation of degree 1 has $\binom{n+1}{r+1}$ invariant subspaces of dimension r.

A degree one foliation is given by a field

$$\mathcal{X} = F_0 \frac{\partial}{\partial Z_0} + F_1 \frac{\partial}{\partial Z_1} + \cdots + F_n \frac{\partial}{\partial Z_n}$$

where F_i is a homogeneous polynomial of degree 1 for all $i = 0, \ldots, n$.

Let us write $F_i = \sum_j a_{ij} Z_j$. Then we associate to $\mathcal{X}$ the matrix of coefficients $B := ((a_{ij}))_{i,j}$ (this matrix will have trace zero because we are taking $\mathcal{X}$ of divergence zero, to ensure uniqueness).

Exercise 14. Prove that the invariant subspaces of dimension r are in correspondence with the dimension $n - r$ invariant subspaces of the transpose B^t.

Since a generic matrix is diagonalizable, the invariant subspaces of dimension $n-r$ are generated by $n-r$ eigenvectors. Therefore, a generic matrix has $\binom{n+1}{n-r}$ invariant subspaces.

Obtain this result from the previous analysis, as follows. In this case the map q from (4.1.3, p. 64) is not generically injective but only finite. It follows from (1.6.5, p. 45), that the degree of q is $c_g(\mathcal{Q} \otimes \mathcal{S}^\vee)$. Now $\mathcal{Q} \otimes \mathcal{S}^\vee = \mathcal{T}\mathbb{G}$, the tangent bundle to the Grassmannian, (see [21, B.5.8. p. 435]). Thus $c_g(\mathcal{T}\mathbb{G})$ can be computed with Bott's formula cf. Theorem (A.7.1, p. 87). We just have to find the number of fixed points for a convenient action of $\mathbb{C}^*$ on $\mathbb{G}$. For a suitable choice of the weights of the action we will find that there is one fixed point in each of the $\binom{n+1}{n-r}$ canonical open sets of $\mathbb{G}$. Alternatively, we could argue invoking Plücker embedding.

4.2. **Foliations with invariant conic.** We set throughout this section $N = N_{1,2,d}$, $V = V_{1,2,d}$ (cf. (2.7, 2.4, p. 48).)

4.2.1. Foliations in $\mathbb{P}^2$ with invariant conic. In this section we find a compactification $\mathbb{Y}_d \subset \mathbb{P}^N$ of the space of foliations of degree $d \geq 2$ in $\mathbb{P}^2$ that have an invariant smooth conic.

Let Y be a parameter space for the family of smooth conics. As we did in the previous sections, we want to describe the incidence variety

$$\widehat{\mathbb{Y}} := \{(\mathcal{C}, \mathcal{X}) \mid \mathcal{C} \text{ is invariant by } \mathcal{X}\} \subset Y \times \mathbb{P}^N$$

as the projective bundle associated to a subbundle $\mathcal{E}$ of the trivial bundle $Y \times V$. If we obtain such description the image of the projection $q_2 : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N$ will be the parameter space for foliations with an invariant conic. But this construction has the drawback that the space of smooth conics is not complete, whereas in order to compute degrees we need vector bundles over complete basis.

Thus we have to compactify the space of smooth conics, for example, allowing singular conics too. The most natural parameter space for conics is $\mathbb{P}^5$ cf. (A.6, p. 86), so we try and use it. But as we will see, when we go on to examine the fibers of $\widehat{\mathbb{Y}}$ over $\mathbb{P}^5$, the singular conics cause problems: the dimensions jump.

What we are going to do to solve this problem is to blowup $\mathbb{P}^5$ along an appropriate subvariety. We will obtain a variety $\mathbb{B}$, a birrational map $\pi : \mathbb{B} \to \mathbb{P}^5$ and a projective bundle $\mathbb{P}(\mathcal{E})$ *over* $\mathbb{B}$ such that it coincides with $\widehat{\mathbb{Y}}$ over the open set of smooth conics. Fortunately $\mathbb{B}$ is a well known variety, the variety of "complete conics" [47].

The construction of $\mathcal{E}$ will not be explicit, so in order to compute the degree of $\mathbb{Y}_d$ we will use Bott's formula (cf. Appendix (A.7, p. 87)). The point is that we only have to know the weights appearing in a decomposition of the fibers of $\mathcal{E}$ over fixed points of an adequate action of $\mathbb{C}^*$ on $\mathbb{B}$, and we will be able to describe these fibers as limits of the fibers over smooth conics.

Finally we present a script for SINGULAR, [27] that implements the calculation of the degree of $\mathbb{Y}_d$.

In Proposition 4.2.13 we find the codimension

$$\operatorname{cod}_{\mathbb{P}^N} \mathbb{Y}_d = 2(d-1)$$

and its degree,

$$\deg \mathbb{Y}_d = \frac{1}{2^5\, 5!}(d-1)\, d\, (d+1)\, (d^7 + 25d^6 + 231d^5 + 795d^4 + 1856d^3 + 2468d^2 + 2256d + 768).$$

4.2.2. The invariance condition for one conic. Fix a generic conic $\mathcal{C} = \mathcal{Z}(P)$ and let $\mathcal{X} \in V$ be a field given by

$$\mathcal{X} = F_0 \frac{\partial}{\partial Z_0} + F_1 \frac{\partial}{\partial Z_1} + F_2 \frac{\partial}{\partial Z_2}.$$

Recalling (2.2.5, p. 50), $\mathcal{C}$ is invariant by $\mathcal{X}$ if and only if there exists a homogeneous polynomial $G \in S_{d-1}$ such that

$$\mathcal{X}(P) := F_0 \frac{\partial P}{\partial Z_0} + F_1 \frac{\partial P}{\partial Z_1} + F_2 \frac{\partial P}{\partial Z_2} = GP.$$

Exercise 15. Prove that $\mathcal{C} = \mathcal{Z}(P)$ is invariant by $\mathcal{X}$ if and only if there exists a unique representative $Y \in S_d \otimes S_1^\vee$ of $\mathcal{X}$ with $Y(P) = 0$.

Therefore, for a fixed conic $\mathcal{C} = \mathcal{Z}(P)$ we may define the linear map

$$\begin{array}{rcl} \varphi_P : S_d \otimes S_1^\vee & \longrightarrow & S_{d+1} \\ X & \mapsto & X(P). \end{array}$$

Observe that $\varphi_P(GR) = 2GP$ for all $G \in S_{d-1}$. Thus φ_P induces a linear map

$$\psi_P : V \to \frac{S_{d+1}}{P \cdot S_{d-1}}.$$

Moreover, $\mathcal{X} \in \operatorname{Ker}\psi_P$ if and only if $\mathcal{C}$ is invariant by $\mathcal{X}$.

These two maps maps fit into the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Ker}\varphi_P \arrow[r] \arrow[d] & \mathrm{Ker}\psi_P \arrow[d] \\
S_{d-1}R \arrow[r] \arrow[d, "\simeq"] & S_d \otimes S_1^\vee \arrow[r] \arrow[d, "\varphi_P"] & V \arrow[d, "\psi_P"] \\
P \cdot S_{d-1} \arrow[r] & S_{d+1} \arrow[r] & \frac{S_{d+1}}{P \cdot S_{d-1}}.
\end{tikzcd}$$

Hence, by the snake lemma we have an isomorphism

$$\mathrm{Ker}\varphi_P \simeq \mathrm{Ker}\psi_P.$$

4.2.3. The incidence variety. Now, let $\mathcal{C}$ vary in the parameter space of conics, $\mathbb{P}^5$. Consider the map of vector bundles over $\mathbb{P}^5$:

$$\varphi : \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S_1^\vee \longrightarrow S_{d+1} \tag{4.5}$$

given by $\varphi(\mathcal{C}, (P, X)) = (\mathcal{C}, X(P))$. As observed above, φ induces a map

$$\psi : \mathcal{O}_{\mathbb{P}^5}(-1) \otimes V \longrightarrow \frac{S_{d+1}}{S_{d-1} \otimes \mathcal{O}_{\mathbb{P}^5}(-1)}.$$

Again this map fits into the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Ker}\varphi \arrow[r, "\simeq"] \arrow[d] & \mathrm{Ker}\psi \arrow[d] \\
\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}R \arrow[r] \arrow[d, "\simeq"] & \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S_1^\vee \arrow[r] \arrow[d, "\varphi"] & \mathcal{O}_{\mathbb{P}^5}(-1) \otimes V \arrow[d, "\psi"] \\
\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1} \arrow[r] & S_{d+1} \arrow[r] & \frac{S_{d+1}}{\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}}.
\end{tikzcd}$$

Twisting by $\mathcal{O}_{\mathbb{P}^5}(1)$ we obtain:

$$\begin{tikzcd}
 & \Theta \arrow[r, "\simeq"] \arrow[d] & \bar{\Theta} \arrow[d] \\
S_{d-1} \cdot R \arrow[r] \arrow[d, "\simeq"] & S_d \otimes S_1^\vee \arrow[r] \arrow[d] & V \arrow[d] \\
S_{d-1} \arrow[r] & \mathcal{O}_{\mathbb{P}^5}(1) \otimes S_{d+1} \arrow[r] & \frac{\mathcal{O}_{\mathbb{P}^5}(1) \otimes S_{d+1}}{S_{d-1}}
\end{tikzcd} \tag{4.6}$$

where

$$\Theta := \mathcal{O}_{\mathbb{P}^5}(1) \otimes \mathrm{Ker}\varphi \simeq \bar{\Theta} := \mathcal{O}_{\mathbb{P}^5}(1) \otimes \mathrm{Ker}\psi.$$

Restricting over the open subset $U \subset \mathbb{P}^5$ of smooth conics, we see that

$$\mathbb{P}(\bar{\Theta}_{|U}) \subset \mathbb{P}^5 \times \mathbb{P}^N$$

is the incidence variety

$$\{(\mathcal{C}, \mathcal{X}) \mid \mathcal{C} \text{ is invariant by } \mathcal{X}\} \subset U \times \mathbb{P}^N.$$

But as we will see soon, $\bar{\Theta}$ is not a vector bundle. In fact its fibers have different dimensions depending on the singularities of the conic.

The isomorphism $\bar{\Theta} \simeq \Theta$ is very usefull, because to describe the jumps in the dimension of the fibers of Θ will be easier.

4.2.4. Θ is not a vector bundle. Let us see why $\mathrm{Im}\varphi$ (and consequently Θ) is not a vector bundle.

Exercise 16. Recall that if

$$X = F_0\frac{\partial}{\partial Z_0} + F_1\frac{\partial}{\partial Z_1} + F_2\frac{\partial}{\partial Z_2} \in S_d \otimes S_1^\vee$$

and $\mathcal{C} = \mathcal{Z}(P)$ then

$$\varphi(\mathcal{C}, (P, X)) = X(P) = F_0\frac{\partial P}{\partial Z_0} + F_1\frac{\partial P}{\partial Z_1} + F_2\frac{\partial P}{\partial Z_2}.$$

Show that $X(P)$ vanishes at the singularities of $\mathcal{C}$, and the dimension of the fibers of $\mathrm{Im}\varphi$ depends on the rank of the conic:

(1) If $\mathcal{C}$ is smooth ($\mathrm{rk}\,\mathcal{C} = 3$) then $\mathrm{rk}\,\varphi_P = \dim S_{d+1} = \binom{d+3}{2}$.
Use that in this case $\mathcal{C}$ is projectively equivalent to $\mathcal{Z}(P)$ with

$$P = Z_0^2 + Z_1^2 + Z_2^2.$$

(2) If $\mathcal{C}$ is the union of two lines ($\mathrm{rk}\,\mathcal{C} = 2$), then $\mathrm{rk}\,\varphi_P = \dim S_{d+1} - 1$. In this case $\mathcal{C}$ is projectively equivalent to $\mathcal{Z}(P)$ with $P = Z_0Z_1$.

(3) If $\mathcal{C}$ is a double line ($\mathrm{rk}\,\mathcal{C} = 1$) then $\mathrm{rk}\,\varphi_P = \dim S_d = \binom{d+2}{2}$. Now $\mathcal{C}$ is projectively equivalent to $\mathcal{Z}(P)$ with $P = Z_0^2$.

4.2.5. The blow-up. Let $r = \binom{d+2}{2}$ denote the minimal rank of φ, and denote by Y_r the scheme defined by the Fitting ideal of φ, generated by the $(r+1) \times (r+1)$-minors of a local representation of φ, cf. (4.5).

The analysis of the dimensions of the fibers above shows that Y_r coincides with the Veronese variety $\mathbb{V}$ of double lines (cf. Appendix (A.6, p. 86)), at least as sets. We are going to blowup $\mathbb{P}^5$ along $\mathbb{V}$ and prove in Lemma 4.2.6 below that this solves our problem.

Let $\mathbb{B}$ denote the blowup of $\mathbb{P}^5$ along $\mathbb{V}$, and $\pi : \mathbb{B} \to \mathbb{P}^5$ the map of blowup (see Appendix A.5, p. 84 and A.6).

Consider the pullback by π of the maps φ and ψ

$$\varphi_{\mathbb{B}} : \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S_1^\vee) \longrightarrow \pi^* S_{d+1}$$

$$\psi_{\mathbb{B}} : \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes V) \longrightarrow \pi^*(\frac{S_{d+1}}{S_{d-1} \otimes \mathcal{O}_{\mathbb{P}^5}(-1)}).$$

The following lemma describes the effect of the blowup in the minors of $\varphi_{\mathbb{B}}$. This result together with Lemma 4.2.8 will be used to prove that blowing up $\mathbb{P}^5$ along $\mathbb{V}$ we obtain a vector bundle.

4.2.6. Lemma. *The $k \times k$ minors of $\varphi_{\mathbb{B}}$ are locally principal for all $k \geq 1$.*

Proof. Let $\varphi_0 : \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_1^\vee \to S_1$ be the universal symmetric map that gives the matrix of the conic. We are blowing-up the ideal of 2×2-minors of φ_0, so we have that the minors of $\varphi_{0\mathbb{B}}$ are locally principal, say generated by t. Thus we can assume that the matrix is locally of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & a_4 \\ 0 & a_4 & a_5 \end{pmatrix}$$

with ideal of 2×2-minors $\langle t, a_4, a_5\rangle$. So t divides a_4, a_5. Performing elementary operations we can assume that the matrix assumes the form

$$A = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & ts \end{smallmatrix}\right).$$

Let us analyze the map $\varphi_{\mathbb{B}}$ at the conic corresponding to A, *i.e.*,

$$\varphi_{\mathcal{C}} : S_d \otimes S_1^\vee \longrightarrow S_{d+1}$$

where $\mathcal{C} = \mathcal{Z}(Z_0^2 + tZ_1^2 + tsZ_2^2)$.

Set $\nu_m := \dim S_m = \binom{m+2}{2}$. Choose a basis for $S_d \otimes S_1^\vee$ as follows: take the first ν_{d+1} vectors as

(ν_d vectors) a basis of $S_d \otimes \dfrac{\partial}{\partial Z_0}$;

($d+1$ vectors) a basis of $\mathrm{Sym}_d(Z_1, Z_2) \otimes \dfrac{\partial}{\partial Z_1}$;

(1 vector) $Z_2^d \dfrac{\partial}{\partial Z_2}$.

Next take

(ν_{d-1} vectors) a basis of $Z_0 S_{d-1} \otimes \dfrac{\partial}{\partial Z_1}$;

($\nu_d - 1$ vectors) a basis of $\dfrac{S_d}{\mathbb{C} \cdot Z_2^d} \otimes \dfrac{\partial}{\partial Z_2}$.

Now we pick the following basis for S_{d+1}:

(ν_d vectors) a basis of $Z_0 S_d$;

($d+1$ vectors) a basis of $Z_1 \mathrm{Sym}_d(Z_1, Z_2)$;

(1 vector) Z_2^{d+1}.

Then the matrix of $\varphi_{\mathcal{C}}$ in this basis looks like

$$A_d = \begin{pmatrix} 2I_{\nu_d} & 0 & 0 & B_1 & B_3 \\ 0 & 2tI_{d+1} & 0 & 0 & B_4 \\ 0 & 0 & 2ts & 0 & 0 \end{pmatrix},$$

where the entries of B_1 are multiples of t, and the entries of B_3, B_4 are multiples of ts. Here I_m stands for the identity matrix of size m.

From this we conclude that the ideals J_i of $i \times i$-minors of A_d are:

$$\begin{cases} J_i & = \langle 1\rangle \quad \text{for } i = 1, \ldots, \nu_d \\ J_{\nu_d + j} & = \langle t^j\rangle \quad \text{for } j = 1, \ldots, d+1 \\ J_{\nu_{d+1}} & = \langle t^{d+2} s\rangle. \end{cases}$$

In particular these minors are principal as we claimed. □

4.2.7. Construction of $\mathcal{E}$. Next, we are going to construct a vector subbundle $\mathcal{E} \subset \mathbb{B} \times V$ over $\mathbb{B}$ which coincides with $\pi^*\bar{\Theta}$ over the open set of smooth conics.

First we prove a technical lemma.

4.2.8. Lemma. *Let R be a local Noetherian domain, and $\varphi : R^n \to R^m$ a homomorphism of free, finitely generated R-modules. Suppose that the ideals $\langle k \times k$ minors of $\varphi\rangle$ are principal for all k. Then $\mathcal{M} := \mathrm{Im}\varphi$ is free.*

Proof. Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

be the $m \times n$ matrix associated to φ with respect to some basis. Thus, the columns of A generate $\mathcal{M}$. By hypothesis for $k = 1$, the ideal of the entries of A is principal:

$$\langle a_{11}, \ldots, a_{ij}, \ldots, a_{mn} \rangle = \langle f \rangle \, .$$

We may assume $f \neq 0$. Let $b_{ij} := \frac{a_{ij}}{f}$. We may suppose $a_{11} = f$. Let $\mathcal{M}'$ be the module generated by the columns of

$$B = \begin{pmatrix} 1 & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} .$$

Equivalently (by elementary operations) $\mathcal{M}'$ is generated by the columns of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & B' \end{smallmatrix}\right)$, where

$$B' = \begin{pmatrix} b_{22} & \cdots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{m2} & \cdots & b_{mn} \end{pmatrix} .$$

Applying induction, we have that $\mathrm{Im}B'$ is free. Thus $\mathcal{M}'$ is free. Since R is a domain we have $\mathcal{M} = f \cdot \mathcal{M}' \simeq \mathcal{M}'$. Hence $\mathcal{M}$ is free. $\square$

4.2.9. Proposition. *There exists a vector bundle $\mathcal{E}$ over $\mathbb{B}$ such that:*

(1) *$\mathcal{E}$ is a subbundle of the trivial bundle $\pi^* V$.*
(2) *$\mathcal{E}$ coincides generically with $\pi^*\bar{\Theta} \simeq \pi^*\Theta$* (cf. 4.6, p. 69).

Proof. By Lemma 4.2.6 and the above Lemma we deduce that $\mathcal{M} := \mathrm{Im}\varphi_{\mathbb{B}}$ is locally free. Therefore we obtain a factorization of $\varphi_{\mathbb{B}} = \iota \circ \widetilde{\varphi}$,

$$\begin{tikzcd} \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S_1^\vee) \arrow[r, "\varphi_{\mathbb{B}}"] \arrow[dr, "\widetilde{\varphi}"'] & \pi^* S_{d+1} \\ & \mathcal{M}, \arrow[u, "\iota"'] \end{tikzcd} \tag{4.7}$$

where $\mathcal{M}$ is a vector bundle.
Observe that $\pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}) = \varphi_{\mathbb{B}}(\pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}R)) \subset \mathcal{M}$. Therefore this factorization induces a factorization of $\psi_{\mathbb{B}} = \bar{\iota} \circ \widetilde{\psi}$,

$$\begin{tikzcd} \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes V) \arrow[r, "\psi_{\mathbb{B}}"] \arrow[dr, "\widetilde{\psi}"'] & \pi^*\left(\dfrac{S_{d+1}}{\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}}\right) \\ & \overline{\mathcal{M}}, \arrow[u, "\bar{\iota}"'] \end{tikzcd}$$

where $\overline{\mathcal{M}} := \dfrac{\mathcal{M}}{\pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1})}$ is a vector bundle. Define

$$\mathcal{E} := \pi^*\mathcal{O}_{\mathbb{P}^5}(1) \otimes \mathrm{Ker}\widetilde{\psi}.$$

It follows that $\mathcal{E}$ is a subbundle of $\mathbb{B} \times V$ that coincides with $\pi^*\bar{\Theta}$ over $\pi^{-1}(U)$, where $U \subset \mathbb{P}^5$ is the open set of smooth conics. Indeed, over $\pi^{-1}(U)$ the map

$\iota : \mathcal{M} \to \pi^* S_{d+1}$ as in (4.7) is an isomorphism. Therefore

$$\overline{\mathcal{M}} \simeq \pi^*(\frac{S_{d+1}}{\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}})$$

and $\mathrm{Ker}\widetilde{\psi} = \mathrm{Ker}\psi_{\mathbb{B}}$ and likewise $\mathcal{E} \simeq \pi^*(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes \mathrm{Ker}\psi_{\mathbb{B}} = \pi^*(\bar{\Theta})$, all over $\pi^{-1}(U)$. $\square$

4.2.10. A parameter space for foliations with invariant conic. Since $\mathcal{E} \subset \mathbb{B} \times V$ is locally split, taking the projectivization yields the following diagram:

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}) \subset \mathbb{B} \times \mathbb{P}^N \arrow[dl, "q_1"'] \arrow[dr, "q"] & \\ \mathbb{B} & & \mathbb{P}^N \end{tikzcd}$$

Define $\mathbb{Y}_d := q(\mathbb{P}(\mathcal{E})) \subset \mathbb{P}^N$. We have that $\mathbb{Y}_d$ is the closure of the variety of foliations with an invariant smooth conic.

In Lemma 4.2.12 below we prove that q is generically injective. Therefore in order to compute $\deg \mathbb{Y}_d$ it is sufficient to calculate $s_5(\mathcal{E})$ (see Lemmas 1.6.5 and 1.6.6, p. 45).

4.2.11. The degree of $\mathbb{Y}_d$.

4.2.12. Lemma. *The projection $q : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N$ is generically injective.*

Proof. By Lemma (1.6.6, p. 45) it is sufficient to find, for $d \geq 2$, a degree d vector field $\mathcal{X}$ with a single invariant conic $\mathcal{C}$ and such that $\mathcal{C}$ is a reduced point in the fiber $q^{-1}(\mathcal{X})$ (equivalently, such that $d_{(\mathcal{C},\mathcal{X})}q$ is injective).

For $d = 2$ we claim that

$$\mathcal{X} = (Z_2Z_0 - Z_1^2)\partial Z_0 - (Z_0Z_1 - Z_2^2)\partial Z_1 + (Z_1Z_2 - Z_0^2)\partial Z_2$$

and $\mathcal{C} = \mathcal{Z}(P)$ with $P = Z_0^2 - Z_1^2 + Z_2^2 = 0$ do the job.

In fact, $\mathcal{X}(P) = 0$. Now, if $\mathcal{C}'$ is another conic invariant by $\mathcal{X}$, say $\mathcal{C}' = \mathcal{Z}(Q)$ with $Q = a_0Z_0^2 + a_1Z_0Z_1 + \cdots + a_5Z_2^2$ then there exists $H = b_0Z_0 + b_1Z_1 + b_2Z_2$ such that $\mathcal{X}(Q) = HQ$. This equality provides a system of linear equations from which we can eliminate b_0, b_1, b_2 (we use Singular, [27], see [16]) to find that $a_0 - a_5 = a_1 = a_2 = a_3 + a_5 = a_4 = 0$. Thus $Q = P$.

To prove that $d_{(\mathcal{C},\mathcal{X})}q$ is injective take for example a tangent vector

$$(a_1Z_0Z_1 + a_2Z_0Z_2 + a_3Z_1^2 + a_4Z_1Z_2 + a_5Z_2^2, v) = (m, v) \in \mathcal{T}_{(P,\mathcal{X})}\mathbb{P}(\mathcal{E}).$$

We have to prove that $v = 0$ implies $a_i = 0$, $\forall\, i = 1, \ldots, 5$. Now

$$(m, v) \in \mathcal{T}_{(P,\mathcal{X})}\mathbb{P}(\mathcal{E}) \Leftrightarrow (P + \varepsilon m, \mathcal{X} + \varepsilon v) \in \mathbb{P}(\mathcal{E})(\mathbb{C}[\varepsilon]).$$

This means that modulo ε^2 we have

$$(\mathcal{X} + \varepsilon v)(P + \varepsilon m) = (P + \varepsilon m)h$$

for some $h = h_1 + \varepsilon h_2$, where h_1, h_2 are polynomials of degree one. Thus,

$$\mathcal{X}(P) + \varepsilon(\mathcal{X}(m) + v(P)) = (P + \varepsilon m)h$$

but $\mathcal{X}(P) = 0$ so we have

$$\varepsilon(\mathcal{X}(m) + v(P)) = h_1P + \varepsilon(h_1m + h_2P).$$

Therefore $h_1 = 0$ and the condition now reads

$$\mathcal{X}(m) + v(P) = h_2 P.$$

So if $v = 0$ this implies $\mathcal{X}(m) = h_2 P$. Performing a simple elimination (*e.g.,* using SINGULAR, [27], see [16]) of coefficients of h_2 we obtain all $a_i = 0$.

For $d = 3$ take

$$\mathcal{X} = (Z_2^2 Z_0 - Z_1^3)\partial Z_0 + (Z_0 Z_1^2 - Z_2^3)\partial Z_1 + (Z_1 Z_2^2 - Z_0^2 Z_2)\partial Z_2$$

As above we can prove that $\mathcal{X}$ has $\mathcal{C} = \mathcal{Z}(Z_0^2 + Z_1^2 + Z_1^2)$ as unique invariant conic and check that $\mathcal{C}$ is a reduced point in $q^{-1}(\mathcal{X})$ (see [16]).

For $d \geq 4$ we will construct, using the example of Jouanolou, a field of degree d with a unique invariant conic.

Recall Jouanolou proves in [32, p. 157] that the field

$$\mathcal{Y} := Z_2^e \partial Z_0 + Z_0^e \partial Z_1 + Z_1^e \partial Z_2$$

has no invariant algebraic subset if $e \geq 2$.

Let P be an irreducible polynomial of degree 2. We claim that $\mathcal{X} := P \cdot \mathcal{Y}$ is a field of degree $d \geq 4$ that has $\mathcal{C} = \mathcal{Z}(P)$ as unique invariant conic (in fact $\mathcal{C}$ is in the singular set of $\mathcal{X}$, but this is sufficient for us). Indeed, $\mathcal{X}(P) = P \cdot \mathcal{Y}(P)$. Now if $\mathcal{C}' = \mathcal{Z}(Q)$ is another conic invariant by $\mathcal{X}$ we have that Q divides $P \cdot \mathcal{Y}(Q)$. If Q is irreducible, this implies that Q divides $\mathcal{Y}(Q)$ *i.e.,* $\mathcal{Z}(Q)$ would be invariant by $\mathcal{Y}$.

If $Q = l_1 l_2$, then l_i divides $P \cdot \mathcal{Y}(l_i)$ (see Definition 2.2.5). Hence $\mathcal{Z}(l_i)$ would be invariant by $\mathcal{Y}$.

To prove that $d_{(P,\mathcal{X})} q$ is injective we argue as in the case $d = 2$. Let

$$(m, v) \in \mathcal{T}_{(P,\mathcal{X})}\mathbb{P}(\mathcal{E}) \Leftrightarrow (P + \varepsilon m, \mathcal{X} + \varepsilon v) \in \mathbb{P}(\mathcal{E})(\mathbb{C}[\varepsilon]),$$

where m is a polynomial of degree two linearly independent of P. Then modulo ε^2 we have:

$$(\mathcal{X} + \varepsilon v)(P + \varepsilon m) = (P + \varepsilon m)h$$

for some $h = h_1 + \varepsilon h_2$, where h_1, h_2 are polynomials of degree $d - 1$. Expanding we obtain

$$\mathcal{X}(P) + \varepsilon(\mathcal{X}(m) + v(P)) = (h_1 + \varepsilon h_2)(P + \varepsilon m).$$

Recalling $\mathcal{X}(P) = P \cdot \mathcal{Y}(P)$, we have

$$P \cdot \mathcal{Y}(P) + \varepsilon(P \cdot \mathcal{Y}(m) + v(P)) = h_1 P + \varepsilon(h_1 m + h_2 P).$$

Therefore $h_1 = \mathcal{Y}(P)$ and the condition reads

$$P \cdot \mathcal{Y}(m) + v(P) = h_2 P + \mathcal{Y}(P)m.$$

Thus, if $v = 0$ we get

$$P\mathcal{Y}(m) = h_2 P + \mathcal{Y}(P)m$$

whence P must divide m (cf. [19]) and this implies $m = 0$.

$\square$

4.2.13. Proposition. *Notation as above, let $\mathbb{Y}_d$ be the compactification for the parameter space of 1-dimensional foliations of degree d on $\mathbb{P}^2$ with an invariant smooth conic. Then the degree of $\mathbb{Y}_d$ is given by*

$$\frac{1}{2^5\,5!}\,(d-1)\,d\,(d+1)\,(d^7 + 25d^6 + 231d^5 + 795d^4 + 1856d^3 + 2468d^2 + 2256d + 768).$$

and its codimension is equal to $2(d - 1)$.

Proof. From Proposition 4.2.9 and definition of Θ we have $\mathrm{rk}(\mathcal{E}) = d(d+2)$. As q is finite we have $\dim \mathbb{Y}_d = \dim \mathbb{P}(\mathcal{E}) = 5 + d(d+2) - 1$. Hence

$$\mathrm{cod}\mathbb{Y}_d = N - \dim \mathbb{Y}_d = (d+1)(d+3) - 1 - (5 + d(d+2) - 1) = 2(d-1)\,.$$

To compute the degree of $\mathbb{Y}_d = \int s_5(\mathcal{E}) \cap [\mathbb{B}]$ we use Bott's formula (A.7.1, p. 87):

$$\int s_5(\mathcal{E}) \cap [\mathbb{B}] = \sum_{p \in \mathbb{B}^T} \frac{s_5^T(\mathcal{E}_p) \cap [p]}{c_5^T(\mathcal{T}_p\mathbb{B})}$$

where $T := \mathbb{C}^*$ acts on $\mathbb{B}$ with isolated fixed points.

The action of T on $\mathbb{B}$ will be induced by an action of T on $\mathbb{P}^2$. With the notation of Appendix A.6, write $\mathbb{P}^2 = \mathbb{P}(F)$, where $F = \mathbb{C}^3$ has basis $\{e_0, e_1, e_2\}$. We begin by considering an action of $T = \mathbb{C}^*$ on F:

$$T \times F \to F$$

given by

$$t \cdot e_i = t^{w_i} e_i \tag{4.8}$$

for some $w_i \in \mathbb{Z}$ to be chosen appropriately.

This action induces an action on $\mathrm{Sym}_2 F^\vee$:

$$T \times \mathrm{Sym}_2 F^\vee \to \mathrm{Sym}_2 F^\vee$$

given by

$$t \cdot Z_i Z_j = t^{-(w_i + w_j)} Z_i Z_j.$$

In this way we obtain an action of T on $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}_2 F^\vee)$. It is easy to see that if we choose the weights in such a way that $\{w_i + w_j,\ \text{with } 0 \le i \le j \le 2\}$ are pairwise distinct, we obtain precisely the following six isolated fixed points in $\mathbb{P}^5$:

$$[1 : 0 : \cdots : 0 : 0], [0 : 1 : \cdots : 0 : 0], \ldots, [0 : 0 : \cdots : 0 : 1].$$

These correspond to the conics defined by the monomials

$$Z_0^2, Z_0Z_1, Z_0Z_2, Z_1^2, Z_1Z_2, Z_2^2.$$

In order to induce an action on $\mathbb{B}$ consider the map (see Appendix A.6):

$$\epsilon : \mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}_2 F^\vee) \dashrightarrow \check{\mathbb{P}}^5 = \mathbb{P}(\mathrm{Sym}_2 \textstyle\bigwedge^2 F^\vee)$$

given by $\epsilon(u) = \bigwedge^2 u$.

Recall that $\mathbb{B} = \overline{\mathrm{Graph}\,\epsilon}$, the closure of the graph of ϵ, and that π denotes the map of blowup $\pi : \mathbb{B} \to \mathbb{P}^5$.

It is easy to see that ϵ is $T-$equivariant. Hence $\mathbb{B} = \overline{\mathrm{Graph}\,\epsilon}$ inherits an action of T. Moreover, if $(A, B) \in \mathbb{B}$ is a fixed point then $A \in \mathbb{P}^5$ is a fixed point, T acts on $\pi^{-1}(A)$ and B is a fixed point for this action.

Therefore, in order to obtain the fixed points in $\mathbb{B}$ we have to find the fixed points on the fiber of π over each fixed point in $\mathbb{P}^5$.

If A is a fixed point with $A \notin \mathbb{V}$ *i.e.*, $A \in \{Z_0Z_1, Z_0Z_2, Z_1Z_2\}$, then $\pi^{-1}(A)$ has just one (fixed) point. So take $A \in \mathbb{V}$ *i.e.*, $A \in \{Z_0^2, Z_1^2, Z_2^2\}$. By Appendix A.5 we have that the exceptional divisor of our blowup is

$$E = \mathbb{P}(\mathcal{N})$$

where $\mathcal{N} := N_{\mathbb{V}}\mathbb{P}^5$ stands for the normal bundle of $\mathbb{V}$ in $\mathbb{P}^5$. Then for $A = Z_0^2$ we have (see (A.18) of Appendix A.6)

$$\pi^{-1}(A) = E_A = \mathbb{P}(\mathbb{C} \cdot Z_0^{2\vee} \otimes \langle Z_1^2, Z_1Z_2, Z_2^2\rangle_{\mathbb{C}}).$$

By our choice of the weights, there are three fixed points in the fiber of each $A \in \mathbb{V}$. For $A = Z_0^2$ these point are

$$Z_0^{2\vee} \otimes Z_1^2, Z_0^{2\vee} \otimes Z_1Z_2, Z_0^{2\vee} \otimes Z_2^2.$$

Summarizing, we have twelve fixed points in $\mathbb{B}$, three of them outside E and nine in E. These fixed points are of three types:

$$Z_iZ_j \text{ with } i \neq j;$$
$$(Z_i^2, Z_i^{2\vee} \otimes Z_jZ_k) \text{ with } j, k \neq i;\ j \neq k;$$
$$(Z_i^2, Z_i^{2\vee} \otimes Z_j^2) \text{ with } i \neq j.$$

The next step is to compute the fibers of $\mathcal{E}$ (see Proposition 4.2.9) over each fixed point.

Suppose that $B \in \mathbb{B}$ is a fixed point. The strategy is to take a curve $B(t) \in \mathbb{B}$ such that

$$\lim_{t\to 0} B(t) = B$$

and such that $A(t) := \pi(B(t)) \in \mathbb{P}^5$ is a curve of smooth conics for $t \neq 0$. Therefore $\mathcal{E}_B$ will be obtained as the limit of $\mathcal{E}_{B(t)} = \pi^*\Theta_{B(t)} = \Theta_{A(t)}$ (notation as in (4.6), p. 69) :

$$\lim_{t\to 0} \Theta_{A(t)} = \mathcal{E}_B$$

This enables us to use the well known space of vector fields of degree d that leave invariant a smooth conic $\mathcal{C} = \mathcal{Z}(G)$ (see [14]), to wit,

$$(\spadesuit) \qquad \Big\{F_{ij}\big(\frac{\partial G}{\partial Z_i}\frac{\partial}{\partial Z_j} - \frac{\partial G}{\partial Z_j}\frac{\partial}{\partial Z_i}\big) \ \mid\ F_{ij} \in S_{d-1}\Big\}$$

modulo multiples of the radial vector field. We will adopt the following notation: for each subset $J := \{v_0, \dots, v_k\} \subset \{Z_0, Z_1, Z_2\}$ we set

$$M_m(J) = \{v_0^m, v_0^{m-1}v_1, \dots, v_k^m\},$$

the canonical monomial basis of $\mathrm{Sym}_m(J)$. We write M_m for $M_m(\{Z_0, Z_1, Z_2\})$.

Set $\mathcal{X}_{i,j} := Z_i\frac{\partial}{\partial Z_i} - Z_j\frac{\partial}{\partial Z_j}$. Notice this is a vector of weight 0, since $t\cdot Z_i = t^{w_i}Z_i$ whereas $t \cdot \frac{\partial}{\partial Z_i} = t^{-w_i}\frac{\partial}{\partial Z_i}$.

We now describe suitable 1-parameter families of smooth conics abutting each type of fixed point.

(1) $B_1 = Z_0Z_1$. We take $A(t) = Z_0Z_1 + tZ_2^2 \in \mathbb{P}^5$. Using the characterization $(\spadesuit)$ we see that the space $\mathcal{E}_{A(t)}$ of vector fields leaving $A(t)$ invariant is given by

$$\Big\{F_{10}(Z_1\tfrac{\partial}{\partial Z_1} - Z_0\tfrac{\partial}{\partial Z_0}),\ F_{20}(Z_1\tfrac{\partial}{\partial Z_2} - 2tZ_2\tfrac{\partial}{\partial Z_0}),\ F_{21}(Z_0\tfrac{\partial}{\partial Z_2} - 2tZ_2\tfrac{\partial}{\partial Z_1}) \ \mid\ F_{ij} \in S_{d-1}\Big\}.$$

Taking limit as $t \to 0$, we find a basis for $\mathcal{E}_{B_1}$:

$$\{F_1\mathcal{X}_{0,1}, F_2\frac{\partial}{\partial Z_2} \ \mid\ F_1 \in M_{d-1}\,,\ F_2 \in M_d \setminus \{Z_2^d\}\}.$$

Clearly this basis consists of T-eigenvectors.

(2) $B_2 = (Z_0^2, Z_0^{2\vee} \otimes Z_1Z_2)$. In this case, we take $A(t) = Z_0^2 + tZ_1Z_2$. With the same procedure as above, we obtain the following basis (of T-eigenvectors) for $\mathcal{E}_{B_2}$:

$$\{F_1Z_0\frac{\partial}{\partial Z_1}, F_2Z_0\frac{\partial}{\partial Z_2}, F_3\mathcal{X}_{1,2} \mid F_1, F_2 \in M_{d-1}\,,\ F_3 \in M_{d-1}(\{Z_1, Z_2\})\}.$$

(3) $B_3 = (Z_0^2, Z_0^{2\vee} \otimes Z_1^2)$. In this case a curve of smooth conics that approximates B_3 is $A(t) = Z_0^2 + tZ_1^2 + t^2Z_2^2$. As before, we obtain the following basis of T-eigenvectors for $\mathcal{E}_{B_3}$:

$$\{F_1Z_0\frac{\partial}{\partial Z_1}, F_2\frac{\partial}{\partial Z_2} \mid F_1 \in M_{d-1}\,,\ F_2 \in M_d \setminus \{Z_2^d\}\}.$$

This concludes the computation of the fibers of $\mathcal{E}$.

Next we obtain, for each fixed point B, a base consisting of T-eigenvectors of $\mathcal{T}_B\mathbb{B}$.

If $B \notin E$ then $\mathcal{T}_B\mathbb{B} \simeq \mathcal{T}_{\pi(B)}\mathbb{P}(\mathrm{Sym}_2 F^\vee)$. For example, for $B_1 = Z_0Z_1$ we have

$$\mathcal{T}_{B_1}\mathbb{B} \simeq \langle Z_0Z_1\rangle^\vee \otimes \langle Z_0^2, Z_0Z_2, \ldots, Z_2^2\rangle.$$

If $B \in E$, then $B = (A, [v])$ with $A \in \mathbb{V}$ and $v \in \mathcal{N}_A$. Now

$$\mathcal{T}_B\mathbb{B} = \mathcal{T}_A\mathbb{V} \oplus \mathrm{Hom}(\mathbb{C}\cdot v, \frac{\mathcal{N}_A}{\mathbb{C}\cdot v}) \oplus \mathbb{C}\cdot v,$$

see (A.15, p. 85).

For $B_2 = (Z_0^2, Z_0^{2\vee} \otimes Z_1Z_2)$ we have:

$$\mathcal{T}_{B_2}\mathbb{B} = \mathcal{T}_{Z_0^2}\mathbb{V} \oplus \langle Z_0^{2\vee} \otimes Z_1Z_2\rangle^\vee \otimes \langle Z_0^{2\vee} \otimes Z_1^2, Z_0^{2\vee} \otimes Z_2^2\rangle \oplus \langle Z_0^{2\vee} \otimes Z_1Z_2\rangle$$

where $\mathcal{T}_{Z_0^2}\mathbb{V} = \langle Z_0^2\rangle^\vee \otimes \langle Z_0Z_1, Z_0Z_2\rangle$.

Similarly, for $B_3 = (Z_0^2, Z_0^{2\vee} \otimes Z_1^2)$ we find

$$\mathcal{T}_{B_3}\mathbb{B} = \mathcal{T}_{Z_0^2}\mathbb{V} \oplus \langle Z_0^{2\vee} \otimes Z_1^2\rangle^\vee \otimes \langle Z_0^{2\vee} \otimes Z_1Z_2, Z_0^{2\vee} \otimes Z_2^2\rangle \oplus \langle Z_0^{2\vee} \otimes Z_1^2\rangle.$$

The explicit calculation in Bott's formula is better left for a script in SINGULAR, [27] (see [19] or [16]).

Note that the above computations of the fibers are performed for fixed d. In order to obtain the polynomial formula in Proposition 4.2.13 we have to interpolate the obtained results. We use Lemma 4.2.14 below which enables us to restrict the computation just for the first sixteen values of $d = 2, \ldots, 17$ and then interpolate the answers obtained.

□

4.2.14. Lemma. *Notation as above, the sum in the right hand side of Bott's formula*

$$\int s_5(\mathcal{E}(d)) \cap [\mathbb{B}] = \sum_{B \in \mathbb{B}^T} \frac{s_5^T(\mathcal{E}(d)_B) \cap [B]}{c_5^T(\mathcal{T}_B\mathbb{B})},$$

is a combination of $w_i's$ cf. (4.8) *with polynomial coefficients in d of degree ≤ 15.*

Proof. For each fixed point B let $\{\xi_1(d), \ldots, \xi_{m(d)}(d)\}$ denote the set of weights of $\mathcal{E}(d)_B$. Since $s_5^T(\mathcal{E}(d)_B)$ is a polynomial in the $T-$equivariant Chern classes

$$\{c_k^T(\mathcal{E}(d)_B) \mid k = 1, \ldots, 5\}$$

it's enough to prove that each

$$c_k^T(\mathcal{E}(d)_B) = \sigma_k(\xi_1(d), \ldots, \xi_{m(d)}(d))$$

is a combination of $w_i's$ with polynomial coefficients in d of degree $\leq 3k$.

Recalling Newton's identities

$$k\sigma_k = \sum_{i=1}^{k}(-1)^{i+1}\sigma_{k-i}p_i$$

where

$$p_k(\xi_1(d), \dots, \xi_{m(d)}(d)) := \sum_{i=1}^{m(d)} \xi_i(d)^k$$

we see that it suffices to prove that $p_k(\xi_1(d), \dots, \xi_{m(d)}(d))$ is a combination of $w_i's$ with polynomial coefficients in d of degree $\leq k+2$.

On the other hand, a careful analysis of the weights appearing in the basis of $\mathcal{E}(d)_B$ at each fixed point shows that these weights can be separated into sets of the form

$$\{\text{weights of } M_e(J)\} \text{ or } \{\text{weights of } M_e(J)\} + w$$

where

$$\begin{cases} w \text{ is a (fixed) combination of } w_i's; \\ e = d, d-1 \text{ and} \\ J = \langle Z_0, Z_1, Z_2\rangle \text{ or} \\ J = \langle Z_i, Z_j\rangle,\ i \neq j\ . \end{cases}$$

From this the reader may be convinced that it's enough to prove the following

Claim: Let $m = m(d,n) := \binom{d+n}{n}$ and $\{\xi_{n,1}(d), \dots, \xi_{n,m}(d)\}$ be the weights associated to a basis $M_d(\{Z_0, \dots, Z_n\})$ of $\mathrm{Sym}_d(\langle Z_0, \dots, Z_n\rangle)$. Then

$$p_k^n(d) := \sum_{i=1}^{m} \xi_{n,i}(d)^k$$

is a combination of $w_i's$ with polynomial coefficients in d of degree $\leq k+n$.

To prove the claim we proceed by induction on $n \geq 1$ and on $k \geq 0$. For $n = 1$,

$$M_d(\{Z_0, Z_1\}) = \{Z_0^d, Z_0^{d-1}Z_1, \dots, Z_0Z_1^{d-1}, Z_1^d\}$$

so that $m(d,1) = d+1$. We have

$$\begin{aligned} p_k^1(d) &= \sum_{i=1}^{d+1} \xi_{1,i}(d)^k = \sum_{i=0}^{d}(iw_0 + (d-i)w_1)^k \\ &= \sum_{i=0}^{d}(i(w_0-w_1) + dw_1)^k = \sum_{i=0}^{d}\sum_{j=1}^{k}\binom{k}{j}(i(w_0-w_1))^j(dw_1)^{k-j} \\ &= \sum_{j=1}^{k}\binom{k}{j}(dw_1)^{k-j}(w_0-w_1)^j\sum_{i=0}^{d} i^j. \end{aligned}$$

The sum $\sum_{i=0}^{d} i^j$ is polynomial in d of degree $j+1$, therefore $p_k^1(d)$ is a combination of $w_i's$ with polynomial coefficients in d of degree $\leq k+1$.

For $k = 0$, we have $p_0^n(d) = m(d,n)$, a polynomial in d of degree n.

For the general case, write the basis $M_d(\{Z_0, \dots, Z_n\})$ in the following form:

$$Z_0M_{d-1}(\{Z_0, \dots, Z_n\}) \cup Z_1M_{d-1}(\{Z_1, \dots, Z_n\}) \cup Z_2M_{d-1}(\{Z_2, \dots, Z_n\}) \cup \cdots \cup \{Z_n^d\}.$$

Then the weights are:

$$w_0 + \{\xi_{n,i}(d-1)\} \cup w_1 + \{\xi_{n-1,i}(d-1)\} \cup w_2 + \{\xi_{n-2,i}(d-1)\} \cup \cdots \cup \{dw_n\}.$$

Hence we can write

$$p_k^n(d) = \sum_{i=1}^{m(d,n)} (\xi_{n,i}(d))^k = \sum_{i=1}^{m(d-1,n)} (w_0 + \xi_{n,i}(d-1))^k +$$
$$\sum_{i=1}^{m(d-1,n-1)} (w_1 + \xi_{n-1,i}(d-1))^k + \sum_{i=1}^{m(d-1,n-2)} (w_2 + \xi_{n-2,i}(d-1))^k + \cdots + (dw_n)^k$$
$$= \sum_{j=0}^{k} \binom{k}{j} w_0^j p_{k-j}^n(d-1) + \sum_{j=0}^{k} \binom{k}{j} w_1^j p_{k-j}^{n-1}(d-1) +$$
$$\sum_{j=0}^{k} \binom{k}{j} w_2^j p_{k-j}^{n-2}(d-1) \ + \cdots + \ (dw_n)^k.$$

By induction we conclude that $p_k^n(d) - p_k^n(d-1)$ is a combination of $w_i's$ with polynomial coefficients in d of degree $\leq k+n-1$, and this implies that $p_k^n(d)$ is a combination of $w_i's$ with polynomial coefficients in d of degree $\leq k+n$.

□

4.3. **Foliations with invariant quadrics.** The varieties of complete quadrics (cf. [47]) can also be employed to construct a compactification of the space of 1-dimensional foliations in $\mathbb{P}^n$ that leave invariant a smooth quadric of arbitrary dimension. For example, in the case of conics and quadrics in $\mathbb{P}^3$ we obtain the following.

4.3.1. Theorem. *Let $\mathbb{Y}_{1,d}$ (resp.$\mathbb{Y}_{2,d}$) denote the closure in $\mathbb{P}^N$ of the variety of 1-dimensional foliations in $\mathbb{P}^3$ that have an invariant smooth conic (resp. quadric surface). Then we have the formulas for the degrees and codimensions,*

(i) $\deg \mathbb{Y}_{1,d} = \frac{4}{8!\,3^2}(d-1)\,d\,\big(207d^{14} + 2763d^{13} + 15447d^{12} + 54395d^{11} + 114847d^{10} + 207891d^9 + 256737d^8 + 225801d^7 + 164937d^6 + 182101d^5 + 38993d^4 + 316221d^3 + 248856d^2 - 118908d - 332640\big)$ *and its codimension is equal to* $4(d-1)$*;*

(ii) $\deg \mathbb{Y}_{2,d} = \frac{1}{9!\,(3!)^9}(d-1)\,d\,(d+1)\,\big(d^{24} + 81d^{23} + 3151d^{22} + 77949d^{21} + 1369333d^{20} + 18084843d^{19} + 185031133d^{18} + 1481854743d^{17} + 9251138050d^{16} + 44737976160d^{15} + 168507293704d^{14} + 503603726976d^{13} + 1212870415960d^{12} + 2353394912904d^{11} + 3628929239056d^{10} + 4249158105672d^9 + 3232639214668d^8 + 413912636928d^7 - 2874493287072d^6 - 3885321416832d^5 - 1115680433472d^4 + 4477695012864d^3 + 8264265366528d^2 + 8139069775872d + 4334215495680\big)$ *and its codimension is equal to* $(d-1)(d+5)$.

Appendix A

A.1. **Vector bundles.** A basic reference for this subject is [44], Chapter VI. A vector bundle $\mathcal{E}$ of rank e over a variety X is a variety $\mathcal{E}$ equipped with a morphism

$$\pi : \mathcal{E} \to X$$

such that

(1) There exists an open covering $\{U_i\}$ of X and isomorphisms
$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^e.$$
(2) Over $U_{ij} := U_i \cap U_j$, the compositions
$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : U_{ij} \times \mathbb{A}^e \to U_{ij} \times \mathbb{A}^e$$
are linear, in the sense that $\varphi_{ij}(x, v) = (x, g_{ij}(x)v)$ with transition functions
$$g_{ij} : U_{ij} \to GL_e(\mathbb{C}).$$
(3) These transition functions are **cocycles**: $g_{ik} = g_{ij}g_{jk}$, $g_{ij}^{-1} = g_{ji}$ and $g_{ii} = 1$.

Conversely, given cocycles $\{g_{ij}\}$ it is possible to define a rank e vector bundle $\mathcal{E}$ whose transition functions are $\{g_{ij}\}$. The morphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^e$ are called **local trivializations**.

A **line bundle** is a vector bundle of rank one.

A.1.1. Morphism of vector bundles. A morphism of vector bundles $\pi : \mathcal{E} \to X$, $\pi' : \mathcal{E}' \to X$ is a morphism $\psi : \mathcal{E} \to \mathcal{E}'$ such that

(1) $\pi' \circ \psi = \pi$ and
(2) if $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^e$ and $\varphi_i' : \pi'^{-1}(U_i) \to U_i \times \mathbb{A}^{e'}$ are local trivializations of $\mathcal{E}, \mathcal{E}'$, then $\psi_i := \psi_{|\pi^{-1}(U_i)}$ fits into the commutative diagram
$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\psi_i"] \arrow[d, "\varphi_i"'] & \pi'^{-1}(U_i) \arrow[d, "\varphi_i'"] \\
U_i \times \mathbb{A}^e \arrow[r, "\psi_i'"'] & U_i \times \mathbb{A}^{e'}
\end{tikzcd}$$
where $\psi_i'(x, v) = (x, \gamma_i(x)v)$ with $\gamma_i : U_i \to \mathrm{Hom}(\mathbb{A}^e, \mathbb{A}^{e'})$ a morphism from U_i to the space of linear maps. Chosing basis, we may think of γ_i as a local matrix representation for ψ.

Since over U_{ij} the diagram below commutes,
$$\begin{tikzcd}
\pi^{-1}(U_{ij}) \arrow[r, "\psi_j"] \arrow[d, "\varphi_j"] \arrow[dd, bend right=60, "\varphi_i"'] & \pi'^{-1}(U_{ij}) \arrow[d, "\varphi_j'"'] \arrow[dd, bend left=60, "\varphi_i'"] \\
U_{ij} \times \mathbb{A}^e \arrow[r, "\psi_j'"'] \arrow[d, "\varphi_{ij}"] & U_{ij} \times \mathbb{A}^{e'} \arrow[d, "\varphi_{ij}'"'] \\
U_{ij} \times \mathbb{A}^e \arrow[r, "\psi_i'"'] & U_{ij} \times \mathbb{A}^{e'}
\end{tikzcd}$$
we have
$$g_{ij}'(x)\gamma_j(x) = \gamma_i(x)g_{ij}(x)\,.$$

A.1.2. Sections of a vector bundle. A section of $\mathcal{E}$ is a morphism $s : X \to \mathcal{E}$ such that $\pi \circ s = \mathrm{id}_X$.

A section s is determined by a collection of functions $s_i : U_i \to \mathbb{A}^e$ such that

$$s_i = g_{ij} s_j$$

in U_{ij}. We have $\varphi_i(s(x)) = (x, s_i(x))$ for $x \in U_i$.

A section of $\mathcal{E}$ determines a morphism of vector bundles that we also denote by s,

$$s : \mathcal{O}_X \to \mathcal{E}.$$

If s is a section of $\mathcal{E}$ as above, the **zero scheme of** s, denoted $\mathcal{Z}(s)$, is defined in each open set U_i by the ideal $\langle s_{i1}, \dots, s_{ie}\rangle$, where $s_i = (s_{i1}, \dots, s_{ie})$ with $s_{ij} \in \mathcal{O}_X(U_i)$.

In fact $\mathcal{Z}(s)$ is the scheme defined by the ideal sheaf image of the map $s^\vee : \mathcal{E}^\vee \to \mathcal{O}_X$ dual of s.

A.1.3. Pull-back of vector bundles. Suppose that $\pi : \mathcal{E} \to X$ is a vector bundle, and let $f : Y \to X$ a morphism. We define a vector bundle over Y, denoted $f^*\mathcal{E}$ as follows.

Take the open covering $\{V_i\}$ of Y, where $V_i := f^{-1}(U_i)$, and glue the patches $\{V_i \times \mathbb{A}^e\}$ along $V_{ij} := V_i \cap V_j$ using the cocycles $h_{ij} := g_{ij} \circ f$, (*i.e.*, by the isomorphism $(y, v) \to (y, g_{ij}(f(y))v)$). The variety obtained in this way has a natural projection to Y, $\rho(y, v) = y$, and $\rho^{-1}(y) = \pi^{-1}(f(y))$, *i.e.*, in the fibers we have $(f^*\mathcal{E})_y = \mathcal{E}_{f(y)}$.

A.1.4. Jet bundles.

We recall the notion of jet bundles associated to a vector bundle. A basic reference is [28, 16.7] and [42]. Here we state without proofs the results that we need in the text.

Let $\mathcal{E}$ be a vector bundle over a smooth projective variety X. For $n \geq 0$ the n-jet bundle associated to $\mathcal{E}$, denoted $\mathcal{P}^n(\mathcal{E})$, is a vector bundle over X whose fiber over $x \in X$ is given by

$$\mathcal{P}^n(\mathcal{E})_x = (\mathcal{O}_X / m_x^{n+1}) \otimes \mathcal{E}_x$$

where m_x is the maximal ideal of the point x.

For each $n \geq 0$ there exist exact sequences:

$$0 \to \mathrm{Sym}_{n+1} \Omega_X \otimes \mathcal{E} \to \mathcal{P}^{n+1}(\mathcal{E}) \to \mathcal{P}^n(\mathcal{E}) \to 0. \tag{A.1}$$

As an example let's analyse the case $n = 0$. We have

$$0 \to \Omega_X \otimes \mathcal{E} \to \mathcal{P}^1(\mathcal{E}) \to \mathcal{E} \to 0. \tag{A.2}$$

Consider the evaluation map

$$ev : X \times H^0(X, \mathcal{E}) \to \mathcal{E}$$

given by $ev(x, s) = (x, s(x))$.

The map ev lifts to a map

$$ev_1 : X \times H^0(X, \mathcal{E}) \to \mathcal{P}^1(\mathcal{E}).$$

Suppose that we are in a neighborhood of $0 \in X$ and that $x = (x_1, \dots, x_m)$ are local coordinates of X.

On the fiber of 0 we have $ev_1(0, s) = (0, s(0) + J_0 s \cdot x)$, where $J_0 s$ is the Jacobian of s at 0. If $s(0) = 0$ then

$$ev_1(s) = J_0 s \cdot x \in \Omega_{X,0} \otimes \mathcal{E}_0 \simeq m/m^2 \otimes \mathcal{E}_0$$

i.e., we retrieve the differential of the section.

In general ev lifts to a map $ev_n : X \times H^0(X, \mathcal{E}) \to \mathcal{P}^n(\mathcal{E})$ given by

$$ev_n(x, s) = (x, s_n(x))$$

where $s_n(x)$ is the Taylor expansion of s truncated in order $n+1$.

We have a commutative diagram

$$\begin{tikzcd} X \times H^0(X,\mathcal{E}) \arrow[r, "ev_n"] \arrow[dr, "ev_{n-1}"'] & \mathcal{P}^n(\mathcal{E}) \arrow[d] \\ & \mathcal{P}^{n-1}(\mathcal{E}) \end{tikzcd} \tag{A.3}$$

A.2. **Cartier Divisors.** General references for this subject are [43] Chapter III and [44] Chapter VI.

A.2.1. Definition. Let X be a scheme. A **Cartier divisor** D on X is given by an affine open cover $\{U_i\}$ of X together with a choice of an invertible element f_i in the total ring of fractions $R(U_i)$ of the coordinate ring $\mathcal{O}_X(U_i)$ such that $f_i {f_j}^{-1}$ is invertible in $\mathcal{O}_X(U_{ij})$, with $U_{ij} = U_i \cap U_j$, $\forall i, j$. Each f_i is said to be a **local equation of D in U_i**.

The data $(\{U_i\}, f_i)$ and $(\{V_\alpha\}, g_\alpha)$ determine the same Cartier divisor if there exists a refinement $\{W_\lambda\}$ of $\{U_i \cap V_\alpha\}$ such that

$$(f_{i\,|W_\lambda})(g_{\alpha\,|W_\lambda})^{-1} \quad \text{is invertible in } \mathcal{O}_X(W_\lambda) \tag{A.4}$$

for all $i = i(\lambda)$, $\alpha = \alpha(\lambda), W_\lambda \subseteq U_i \cap V_\alpha$.

For each Cartier divisor on a scheme X, we are given a collection of local equations $f_x \in R(\mathcal{O}_{X,x})$, $\forall x \in X$ with the following property. For each $x \in X$, there exists an affine neighborhood U_x together with some $\widetilde{f} \in R(U_x)$ such that f_y is the image of $\widetilde{f}$ in $R(U_y)$ for all $y \in U_x$. Two such collections $\{f_x\}, \{g_x\}$ define the same Cartier divisor if and only if for all x we have that $f_x {g_x}^{-1}$ lies in $\mathcal{O}^\star_{X,x}$, the subgroup of invertible elements. Put in other words, a Cartier divisor is an element of $H^0(X, R^\star/\mathcal{O}^\star)$.

A Cartier divisor is said to be **effective** if it admits a representation by local equations $(\{U_i\}, f_i)$ such that f_i is a regular function, *i.e.*, f_i lies in $\mathcal{O}_X(U_i)$ for all i. This is the same as a closed subscheme locally defined by a nonzero divisor.

A Cartier divisor $D = (\{U_i\}, f_i)$ is said to be **principal** if $f_i = f_j$ in U_{ij}, $\forall i, j$. In other words, the given local equations are compatible along the intersection, thereby yielding a global section f of the subsheaf $R^\star_X$ of invertible elements of the sheaf of total ring of fractions, so that we may also write $D = (\{X\}, f)$, a single equation.

A.2.2. Example. *Let $X = \mathbb{P}^n$ and let $F(Z_0, \dots, Z_n)$ be a nonzero homogeneous polynomial of degree m. Let U_i be the standard affine open subset complementary of the hyperplane $Z_i = 0$. The coordinate ring of U_i is the polynomial ring in the indeterminates $Z_0/Z_i, \dots, Z_n/Z_i$. Put*

$$f_i = Z_i^{-m} F = F(Z_0/Z_i, \dots, Z_n/Z_i) \in \mathcal{O}_X(U_i).$$

Then (U_i, f_i), $i = 0, \dots, n$ is an effective Cartier divisor. It is equal to the hypersurface defined by F.

A.2.3. Definition. Let $D = (\{U_i\}, f_i)$ be a Cartier divisor on X. The **cycle associated** to D is

$$[D] = \sum \mathrm{ord}_V(D) \cdot V,$$

where the sum is taken over the subvarieties of codimension one and the coefficient is defined by

$$\mathrm{ord}_V(D) = \mathrm{ord}_{V_i}(f_i) \tag{A.5}$$

with $V_i = U_i \cap V \neq \emptyset$.

A.2.4. Definition. Let $D = (\{U_i\}, f_i)$ be a Cartier divisor on X. We write $\mathcal{O}_X(D)$ for the *line bundle associated* to D, defined by the transition functions $f_{ij} = f_i f_j^{-1}$ on U_{ij} (cf.[43], p.270).

Explicitly, $\mathcal{O}_X(D)$ is the scheme over X obtained by glueing. One takes the disjoint union

$$\coprod U_i \times \mathbb{A}^1$$

and identify pairs

$$(x, v) \in U_i \times \mathbb{A}^1,\ (y, w) \in U_j \times \mathbb{A}^1$$

if and only if

$$x = y \in U_{ij} \quad \text{and} \quad v = f_{ij}(x)w.$$

In other words, we glue the open affine subsets $U_i \times \mathbb{A}^1, U_j \times \mathbb{A}^1$ identifying the open subsets $U_{ij} \times \mathbb{A}^1 \subseteq U_i \times \mathbb{A}^1$, $U_{ij} \times \mathbb{A}^1 \subseteq U_j \times \mathbb{A}^1$ via the isomorphism $A[T] \simeq A[T]$ defined by $T \mapsto f_{ij} \cdot T$, where A denotes the coordinate ring of U_{ij}.

A.2.5. Proposition. *Let $\mathcal{L} \to X$ be a line bundle over a variety. Then there exists a Cartier divisor D on X such that $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{L}$.*

Proof. Let $\{U_i\}$ be an affine open cover of X and let $f_{ij} \in \mathcal{O}_X(U_{ij})^\star$ be transition functions for $\mathcal{L}$. Since X is a variety, each coordinate ring $\mathcal{O}_X(U_{ij})$ is a domain, contained in the function field $R(X) = R(U)$ for any open subset $U \neq \emptyset$. Fix an index i_0, and write it 0 for short. Set $f_i = f_{i0}$. It is clear that $(\{U_i\}, f_i)$ defines a Cartier divisor D. Furthermore, the associated line bundle $\mathcal{O}_X(D)$ is given by the transition functions $f_i f_j^{-1} = f_{i0} f_{j0}^{-1} = f_{ij}$, whence $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{L}$. □

A.2.6. Remark. The result above does not hold for arbitrary schemes, cf. Hartshorne, [30].

A.3. **Projective bundles.** Associated to a vector bundle $\mathcal{E}$ we have a projective bundle $\mathbb{P}(\mathcal{E})$. It is obtained by replacing the vector space fibers of $\mathcal{E}$, all isomorphic to $\mathbb{A}^e$, by the projective space $\mathbb{P}(\mathbb{A}^e) \simeq \mathbb{P}^{e-1}$. See [44, Chapter VI p.73] and [21, Appendix B.5].

Explicitly, given a vector bundle defined by transition functions $\{g_{ij}\}$, we glue the patches $\{U_i \times \mathbb{P}^{e-1}\}$ along $U_{ij} \times \mathbb{P}^{e-1}$ using the linear isomorphisms g_{ij}'s. We glue $U_i \times \mathbb{P}^{e-1}$ with $U_j \times \mathbb{P}^{e-1}$ with the isomorphism $(x, [v]) \mapsto (x, [g_{ij} v])$, for $x \in U_{ij}$.

We have a projection

$$p : \mathbb{P}(\mathcal{E}) \to X \text{ such that } p(\varphi_i^{-1}(x, [v])) = x.$$

This map is proper.

Consider $p^*\mathcal{E}$, it is a vector bundle over $\mathbb{P}(\mathcal{E})$, whose fiber over $(x, [v])$ is $\mathcal{E}_x$.

In analogy with the tautological line bundle of $\mathbb{P}^n$, there exist a **tautological** vector subbundle $\mathcal{O}_{\mathcal{E}}(-1)$ of $p^*\mathcal{E}$, whose fiber over $(x,[v]) \in \mathbb{P}(\mathcal{E})$ is $\mathbb{C}v$. We have

$$\mathcal{O}_{\mathcal{E}}(-1) \longrightarrow p^*\mathcal{E}$$

which is nonzero on every fiber. The above correspond to a section

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})} \longrightarrow p^*\mathcal{E} \otimes \mathcal{O}_{\mathcal{E}}(1).$$

The cokernel is the **relative tangent bundle** of $\mathbb{P}(\mathcal{E})$ over X:

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \longrightarrow p^*\mathcal{E} \otimes \mathcal{O}_{\mathcal{E}}(1) \longrightarrow \mathcal{T}_{\mathbb{P}(\mathcal{E})/X} \to 0. \tag{A.6}$$

A.4. **Grassmannians.** For the definition and first properties of Grassmannians consult ([29] Lecture 6) or [43] and [44]). Let $\mathbb{G} = \mathbb{G}(k,n)$ denote the variety parametrizing projective k-planes of $\mathbb{P}^n$ (equivalently $\mathbb{G}$ parametrizes vector $(k+1)$-planes of $\mathbb{C}^{n+1}$.) We have

$$\dim \mathbb{G}(k,n) = (k+1)(n-k)\,.$$

There exists a tautological exact sequence of fiber bundles over $\mathbb{G}$,

$$0 \to \mathcal{S} \to \mathbb{G} \times \mathbb{C}^{n+1} \to \mathcal{Q} \to 0 \tag{A.7}$$

where $\mathcal{S}$ is of rank $k+1$ and $\mathcal{Q}$ is of rank $n-k$. Explicitly, if $W \in \mathbb{G}$ is the projectivization of a $k+1$-plane $\Lambda \subset \mathbb{C}^{n+1}$,*i.e.*, $W = \mathbb{P}(\Lambda)$, then the fiber of $\mathcal{S}$ over W is $\mathcal{S}_W = \Lambda$ (respectively, the fiber of $\mathcal{Q}$ is $\mathbb{C}^{n+1}/\Lambda$).

If we dualize the sequence (A.7), we obtain

$$0 \to \mathcal{Q}^\vee \to \mathbb{G} \times \check{\mathbb{C}}^{n+1} \to \mathcal{S}^\vee \to 0. \tag{A.8}$$

The fiber of $\mathcal{Q}^\vee$ over W is the subspace of $\check{\mathbb{C}}^{n+1}$ generated by the equations defining W.

In the case $k = 0$, we have $\mathbb{G}(0,n) = \mathbb{P}^n$, and the tautological bundle is $\mathcal{S} = \mathcal{O}_{\mathbb{P}^n}(-1)$.

The projective bundle $\mathbb{P}(\mathcal{S})$ is the *universal k-plane*:

$$\mathbb{P}(\mathcal{S}) = \{(W,p) \in \mathbb{G} \times \mathbb{P}^n \mid p \in W\}.$$

We have projection maps

$$\begin{tikzcd} & \mathbb{P}(\mathcal{S}) \subset \mathbb{G} \times \mathbb{P}^n \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{G} & & \mathbb{P}^n \end{tikzcd} \tag{A.9}$$

Observe that $p_2^*\mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{O}_{\mathbb{P}(\mathcal{S})}(-1)$.

A.5. **Blowup.** In this section we present without proofs some basic facts about the blowup of a scheme along a subscheme. A reference for this subject is [21, Appendix B.6] or [43, Chapter II].

Let X be a closed subscheme of a scheme Y, defined by an ideal sheaf $\mathcal{J}$. Then the blowup of Y along X, denoted $\widetilde{Y}$ is defined by

$$\widetilde{Y} := \operatorname{Proj}\Big(\bigoplus_{n\geq 0} \mathcal{J}^n\Big).$$

Denote by $\pi : \widetilde{Y} \to Y$ the projection and set $E := \pi^{-1}(X)$. Then E is a Cartier divisor, called the **exceptional divisor**. Moreover, π restricted to $\widetilde{Y} \setminus E$ is an isomorphism onto $Y \setminus X$.

Suppose that the embedding of X in Y is regular of codimension d. Then we have

$$E = \mathbb{P}(\mathcal{N})$$

with projection $\eta : E \to X$, where $\mathcal{N} = \mathcal{N}_X Y$, stands for the normal bundle. Moreover,

$$\mathcal{N}_E\widetilde{Y} = \mathcal{O}_{\widetilde{Y}}(E)_{|E} = \mathcal{O}_{\mathcal{N}}(-1). \tag{A.10}$$

Next we want to compute the fiber of $T\widetilde{Y}$ over a point $y \in E$. We have the following exact sequence

$$0 \to \mathcal{T}E \to \mathcal{T}\widetilde{Y}_{|E} \to \mathcal{N}_E\widetilde{Y} \to 0.$$

Therefore

$$\mathcal{T}_y\widetilde{Y} = \mathcal{T}_y E \oplus \mathcal{O}_{\mathcal{N}}(-1)_y \tag{A.11}$$

though not canonically. However, if $y \in Y$ happens to be a fixed point of some $\mathbb{C}^*-$action on Y that leaves X invariant, the above decomposition is unique as $\mathbb{C}^*-$modules.

If $y = (x, [v])$, with $x \in X$ and $v \in \mathcal{N}_x$, then

$$\mathcal{O}_{\mathcal{N}}(-1)_y = \mathbb{C} \cdot v. \tag{A.12}$$

In order to compute $\mathcal{T}_y E$, we observe that

$$\mathcal{T}_y E = \mathcal{T}_x X \oplus \mathcal{T}_{[v]}\mathbb{P}(\mathcal{N}_x). \tag{A.13}$$

But $\mathcal{T}_{[v]}\mathbb{P}(\mathcal{N}_x)$ is the fiber over y of the relative tangent bundle of $\mathbb{P}(\mathcal{N})$ over X, which is defined by the following (Euler) exact sequence (see [21, B.5.8.]):

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{N})} \to \eta^*\mathcal{N} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N})}(1) \to \mathcal{T}_{\mathbb{P}(\mathcal{N})/X} \to 0.$$

Moreover, from this we deduce that $\mathcal{T}_{\mathbb{P}(\mathcal{N})/X} = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(\mathcal{N})}(-1), \mathcal{Q})$ where $\mathcal{Q}$ is the universal quotient bundle of $\mathbb{P}(\mathcal{N})$. Thus

$$\mathcal{T}_{[v]}\mathbb{P}(\mathcal{N}_x) = \mathrm{Hom}(\mathbb{C} \cdot v, \frac{\mathcal{N}_x}{\mathbb{C} \cdot v}). \tag{A.14}$$

Putting together (A.11), (A.12), (A.13) and (A.14) we obtain

$$\mathcal{T}_y\widetilde{Y} = \mathcal{T}_x X \oplus \mathrm{Hom}(\mathbb{C} \cdot v, \frac{\mathcal{N}_x}{\mathbb{C} \cdot v}) \oplus \mathbb{C} \cdot v. \tag{A.15}$$

As a final remark observe that since π is an isomorphism from $\widetilde{Y} \setminus E$ onto $Y \setminus X$ we have, for a point $y \in \widetilde{Y} \setminus E$,

$$\mathcal{T}_y\widetilde{Y} = \mathcal{T}_{\pi(y)}Y.$$

A.6. **Complete conics.** In this subsection we review some results about conics and the space of complete conics. References for this topic are [29], [47].

Let F denote the vector space $\mathbb{C}^3$, $\mathbb{P}^2 = \mathbb{P}(F)$. A conic is given by a nonzero symmetric map $u : F \to F^\vee$ modulo non-zero multiples, *i.e.,* an element of $\mathbb{P}(\mathrm{Sym}_2(F^\vee)) = \mathbb{P}^5$.

The rank of a conic is by definition the rank of the map u. It defines two distinguished subvarieties in $\mathbb{P}(\mathrm{Sym}_2(F^\vee))$. The first one is the locus of double lines, corresponding to the maps with $\mathrm{rk}\, u = 1$. We denote it by $\mathbb{V}$. The locus of singular conics (the maps with $\mathrm{rk}\, u \leq 2$) is denoted by $\mathbb{V}_2$. We have that $\mathbb{V}_2$ is the (cubic) hypersurface defined by $\det(u) = 0$ and $\mathbb{V}$ is the Veronese surface, given by the image of

$$\nu_2 : \mathbb{P}(F^\vee) \to \mathbb{P}(\mathrm{Sym}_2 F^\vee)$$

where $\nu_2([a_0 : a_1 : a_2]) = [a_0^2 : 2a_0a_1 : \cdots : a_2^2]$ *i.e.,* ν_2 sends a line $L := a_0Z_0 + a_1Z_1 + a_2Z_2$ to $L^2 := a_0^2Z_0^2 + 2a_0a_1Z_0Z_1 + \cdots + a_2^2Z_2^2$.

Next we compute the tangent and normal spaces of the Veronese variety in $\mathbb{P}^5$. Suppose that the double line we are considering is Z_0^2. Then a vector $v = (a_1, a_2, \ldots, a_5)$ is in $\mathcal{T}_{Z_0^2}\mathbb{V}$ if and only if

$$Z_0^2 + \varepsilon(a_1Z_0Z_1 + a_2Z_0Z_2 + \cdots + a_5Z_2^2) \in \mathbb{V}(\mathbb{C}[\varepsilon])$$

i.e., if the matrix representing this conic has all 2×2–minors equal to zero over the ring $\mathbb{C}[\varepsilon]$, $\varepsilon^2 = 0$. This matrix reduces (after some elementary operations) to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon a_3 & \varepsilon a_4 \\ 0 & \varepsilon a_4 & \varepsilon a_5 \end{pmatrix}.$$

So it is clear that all 2×2 minors of A vanish if and only if $a_3 = a_4 = a_5 = 0$. It follows that

$$\mathcal{T}_{Z_0^2}\mathbb{V} = \mathbb{C} \cdot Z_0^{2\vee} \otimes \langle Z_0Z_1, Z_0Z_2\rangle_{\mathbb{C}} \subset \mathcal{T}_{Z_0^2}\mathbb{P}^5 = (\mathcal{O}_{\mathbb{P}^5}(1) \otimes Q)_{Z_0^2}. \tag{A.16}$$

and

$$(\mathcal{O}_{\mathbb{P}^5}(1) \otimes Q)_{Z_0^2} = \mathbb{C} \cdot Z_0^{2\vee} \otimes \langle Z_0Z_1, Z_0Z_2, Z_1^2, Z_1Z_2, Z_2^2\rangle_{\mathbb{C}}. \tag{A.17}$$

Consequently the normal to $\mathbb{V}$ in $\mathbb{P}^5$ is:

$$\mathcal{N}_{Z_0^2} = \mathbb{C} \cdot Z_0^{2\vee} \otimes \langle Z_1^2, Z_1Z_2, Z_2^2\rangle_{\mathbb{C}}. \tag{A.18}$$

In fact, in [47, Proposition 4.4.] is proved that if we consider the Grassmannian of lines in $\mathbb{P}^2$ with tautological sequence:

$$\mathcal{S}_2 \to \mathbb{G} \times F \to \mathcal{Q}_2$$

where $\mathcal{S}_2$ has rank 2, then

$$\mathcal{N}_{\mathbb{V}}\mathbb{P}^5 = (\mathcal{O}_{\mathbb{P}^5}(1) \otimes \mathrm{Sym}_2(\mathcal{S}_2^\vee))_{|\mathbb{V}}.$$

This may clarify the description (A.18).

The Gauss map associates to each point on a conic $\mathcal{C}$ its tangent line. If a conic $\mathcal{C}$ is smooth, this map is an isomorphism and the dual $\mathcal{C}^*$ is again a smooth conic in $\check{\mathbb{P}}^2$, also referred to as the envelope of tangent lines to $\mathcal{C}$. It is the restriction to the conic of the map $\mathbb{P}(F) \to \mathbb{P}(F^\vee)$ induced by the linear map u. But there is no well defined tangent line at a singular point of the conic. In order to produce a well defined envelope for every conic, H. Schubert ([45]) introduced the variety of "complete conics". This variety is a compactification of the variety of smooth conics, different from $\mathbb{P}^5$. Let us explain how this compactification is obtained. Consider the rational map

$$\epsilon : \mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}_2 F^\vee) \dashrightarrow \check{\mathbb{P}}^5 = \mathbb{P}(\mathrm{Sym}_2 \overset{2}{\wedge} F^\vee)$$

given by $\epsilon(u) = \overset{2}{\wedge} u$. This map restricted to the open set of smooth conics is a bijection that sends a conic to its dual conic.

The variety of complete conics is the blowup $\mathbb{B}$ of $\mathbb{P}^5$ along $\mathbb{V}$. In [47, p. 210] it is proved that $\mathbb{B}$ is embedded in $\mathbb{P}^5 \times \check{\mathbb{P}}^5$ as $\mathbb{B} = \overline{\mathrm{Graph}\,\epsilon}$, closure of the graph of ϵ.

A.7. **Bott's Formula.** In this section we explain Bott's equivariant formula. A reference for this subject in the general case is [40] and the bibliography therein.

Let X be a smooth complete variety of dimension n, and let $T = \mathbb{C}^*$ act on X with isolated fixed points. Write X^T for the set of fixed points.

Let $\mathcal{E}$ be a T-equivariant vector bundle over X of rank r.

If $p(c_1, \ldots, c_r)$ is a weighted homogeneous polynomial of total degree n with rational coefficients, where $\deg c_i = i$, then

$$p(c_1(\mathcal{E}), \ldots, c_r(\mathcal{E})) \cap [X]$$

is a zero cycle in X. Bott's formula expresses the degree of this zero cycle in terms of data given by the induced action of T on the fibers of $\mathcal{E}$ and of the tangent bundle $\mathcal{T}X$ over the fixed points of the action. Below is an outline for its usage.

Let $p \in X^T$ be a fixed point. The torus T acts on the fiber $\mathcal{E}_p$ and (as T is semisimple) we have a complete decomposition of $\mathcal{E}_p$ into T-eigenspaces, with certain weights $\xi_i \in \mathbb{Z}$:

$$\mathcal{E}_p = \oplus_{i=1}^r \mathcal{E}_p^{\xi_i}$$

with

$$\mathcal{E}_p^{\xi_i} = \{v \in \mathcal{E}_p \mid t \cdot v = t^{\xi_i} v, t \in T\}.$$

Set

$$c_i^T(\mathcal{E}_p) := \sigma_i(\xi_1, \ldots, \xi_r),$$

where σ_i denotes the i-th elementary symmetric polynomial:

$$\sigma_1 = \sum \xi_j, \ \sigma_2 = \sum_{i<j} \xi_i \xi_j, \ \ldots, \ \sigma_r = \xi_1 \cdots \xi_r .$$

Set $p^T(\mathcal{E}_p) = p(c_1^T(\mathcal{E}_p), \ldots, c_r^T(\mathcal{E}_p))$. Here the magic comes:

A.7.1. Theorem. (Bott's formula)

$$\int p(c_1(\mathcal{E}), \ldots, c_r(\mathcal{E})) \cap [X] = \sum_{p \in X^T} \frac{p^T(\mathcal{E}_p)}{c_n^T(\mathcal{T}_p X)}.$$

□

It is a nice fact that the integer appearing in the left hand side is obtained as a sum of *rational* numbers!

References

[1] A. Altman and S. Kleiman. *Foundations of the Theory of Fano Schemes.* Compositio Math. **34**, 3–47 1977.

[2] O. Calvo-Andrade. *Irreducible components of the space of holomorphic foliations.* Math. Ann. 299 1994, no. 4, 751–767.

[3] O. Calvo-Andrade, D. Cerveau, L. Giraldo and A. Lins Neto. *Irreducible components of the space of foliations associated to the affine Lie algebra.* Ergodic Theory Dynam. Systems 24, no. 4, 987–1014, 2004.

[4] C. Camacho and A. Lins Neto. *Teoria Geométrica das Folheações.* IMPA,1979.

[5] D. Cerveau and A. Lins Neto. *Holomorphic Foliations in $\mathbb{P}^2$ having an Invariant Algebraic Curve.* Annales de l'Institut Fourier 41 fasc. 4, 883-904, 1991.

[6] ______. *Irreducible Components of the Space of Holomorphic Foliations of degree two in $\mathbb{P}^n$, $n \geq 3$* The Annals of Mathematics, Second Series, vol. 143, No 3, pp. 577-612, 1996.

[7] C. Christopher, J. Llibre and J.V. Pereira. *Multiplicity of Invariant Algebraic Curves in Polynomial Vector Fields* Pacific Journal of Mathematics Vol. 229, No. 1, p.63-117, 2007.

[8] G. N. Costa, *Holomorphic foliations by curves on $\mathbb{P}^3$ with non-isolated singularities*, Ann. F. Sciences de Toulouse, Sér. 6, 15 no. 2, p. 297–321, 2006.

[9] S.C. Coutinho and J.V. Pereira. *On the density of algebraic foliations without algebraic invariant sets.* J. Reine Angew. Math., 594. 2006.

[10] F. Cukierman and J.V. Pereira. *Stability of Holomorphic Foliations with Split Tangen Sheaf.* American Journal of Mathematics 130.2. 413-439, 2008.

[11] F. Cukierman, J.V. Pereira and I. Vainsencher. *Stability of Foliations induced by rational maps.* Annales de la Faculté des Sciences de Toulouse. 2007.

[12] F. Cukierman, M. Soares and I. Vainsencher. *Singularities of logarithmic foliations.* Compositio Mathematica. vol. 142, p 131-142, 2006.

[13] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, 150. New York Springer-Verlag, 1994.

[14] E. Esteves. *The Castelnuovo-Mumford regularity of an integral variety of a vector field on projective space*, Math. Res. Lett. **9**, 1-15.

[15] M. Falla Luza. *Global Geometry of Second Order Differential Equations.* Doctoral Thesis, IMPA. `http://www.impa.br/opencms/pt/ensino/downloads/teses_de_doutorado/teses_2010/Hernan_Maycol_Falla_Luza.pdf`. 2010

[16] V. Ferrer. *Aspectos enumerativos de Folheações Holomorfas.* Doctoral Thesis, UFMG. 2010. `http://www.mat.ufmg.br/intranet-atual/pgmat/TesesDissertacoes/uploaded/Tese024.pdf`.

[17] V. Ferrer and I. Vainsencher. *Polynomial vector fields with algebraic trajectories.* Commutative algebra and its connections to geometry, pp. 71-85, Contemp. Math., 555, Amer. Math. Soc., Providence, RI, 2011.

[18] ______. *Degenerate singularities of one dimensional foliations.* Commentarii Mathematici Helvetici. Vol 88. Issue 2, pp.305-321, 2013

[19] ______. *Scripts.* `http://www.mat.ufmg.br/~israel/Publicacoes/Vivis/index.html`. 2010.

[20] ______. *Enumerative Aspects of Holomorphic Foliations.* Monografas del IMCA N. 53, v. 1. 77p. Lima. 2010.

[21] W. Fulton. *Intersection Theory.* Springer-Verlag. New York. 1985.

[22] L. Göttsche, *A conjectural generating function for numbers of curves on surfaces*, Comm. Math. Phys. 196, 523–533, 1998. eprint alg-geom/9711012

[23] X. Gómez-Mont. *Holomorphic foliations in ruled surfaces.* Trans. Amer. Math. Soc., 312(1):179-201, 1989.

[24] ______. *On the spaces of polynomial vector fields modulo projectivities*, Proc. Cong. Dyn. Syst., Trieste 1988, Pitman 1990, 112-127.

[25] ______. *On foliations in $\mathbb{P}^2$ tangent to an algebraic curve*, Proc. Cong. Alg. Geom., Cimat, 1989; Aportaciones Matematicas, Investigacion 5 , 87-99, 1992.

[26] X. Gómez-Mont and L. Ortíz-Bobadilla. *Sistemas dinámicos holomorfos en superficies.* Sociedad Matemática Mexicana, México City, 1989.

[27] G.-M. Greuel, G. Pfister and H. Schönemann, Singular 3-1-1, A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern, 2010. `http//www.singular.uni-kl.de`.

[28] A. Grothendieck and J. Dieudonné. *Éléments de Géométrie Algébrique IV-4*. Publ. Math. IHES, **32**, 5–361, 1967.
[29] J. Harris. *Algebraic Geometry. A First Course*. Graduate Texts in Mathematics. 133. New York - Heidelberg - Berlin: Springer-Verlag, 1992.
[30] R. Hartshorne. *Ample subvarieties of algebraic varieties*, LNM 156 Springer-Verlag, 1970.
[31] ______. *Algebraic Geometry*. Graduate Texts in Mathematics. 52. New York - Heidelberg - Berlin: Springer-Verlag, 1977.
[32] J.P. Jouanolou. *Equations de Pfaff algébriques*. Lectures Notes in Mathematics, **708**. Springer-Verlag, 1979.
[33] ______. *Hypersurfaces solutions d´une équation de Pfaff analytique*. Math. Ann. 232(3), 239-245, 1978.
[34] S. Katz and S. A. Stromme. SCHUBERT: *A Maple package for Intersection Theory*. 2001. Available at `http://stromme.uib.no/home/schubert/`.
[35] S. L. Kleiman and R. Piene. *Node polynomials for families: methods and applications*. Math. Nachr. 271, 69–90, 2004.
[36] D. Laksov. *Complete linear maps*. Ark. Mat. 26, 231-263, 1988.
[37] A. Lins Neto. *Algebraic Solutions of Polynomial Differential Equations and Foliations in Dimension 2*. Holomorphic Dynamics, Springer Lecture Notes In Mathematics, v. 1345, pp. 192-232. 1988.
[38] ______. *Componentes Irredutíveis dos Espaços de Folheações*. 26 Colóquio Brasileiro de Matemática, IMPA, 2007.
[39] A. Lins Neto. and B. Scardua. *Folheações Algébricas Complexas*. 21 Colóquio Brasileiro de Matemática, IMPA, 1997.
[40] A. Meireles Araújo and I. Vainsencher. *Equivariant intersection theory and Bott's residue formula*. 16th School of Algebra, Part I. Brasília, 2000. Mat. Contemp. 20, 2001.
[41] J. V. Pereira. *Vector Fields, Invariant Varieties and Linear Systems*, Annales de L'Institut Fourier, 51 no.5, 1385-1405, 2001.
[42] R. Piene. *Some formulas for a surface in* P^3. Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977), pp. 196–235, Lecture Notes in Math., 687, Springer, Berlin, 1978.
[43] I. Shafarevich. *Basic Algebraic Geometry I*. Second Edition, New York - Heidelberg - Berlin. Springer-Verlag, 1994.
[44] ______. *Basic Algebraic Geometry 2*. Second Edition, New York - Heidelberg - Berlin. Springer-Verlag, 1997.
[45] H. Schubert. *Kalkül der abzählenden Geometrie*, reprint, Springer-Verlag. 1979.
[46] I. Vainsencher. *Classes características em Geometria Algébrica*. 15 Colóquio Brasileiro de Matemática, IMPA, 1985. *Characteristic Classes in Algebraic Geometry*, available at `http://www.mat.ufmg.br/~israel/Ensino/int.pdf`.
[47] ______. *Schubert calculus for complete quadrics*. Enumerative geometry and classical algebraic geometry (Nice, 1981), 199–235, Progr. Math., 24, Birkhäuser, Boston, Mass. 1982.

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