ON CLOSED ORBITS FOR
TWISTED AUTONOMOUS TONELLI LAGRANGIAN FLOWS

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ABSTRACT. These lecture notes were prepared in occasion of a mini-course given by the author at the "CIMPA Research School - Hamiltonian and Lagrangian Dynamics" (10–19 March 2015 - Salto, Uruguay). In this series of talk we illustrated some techniques to prove the existence of periodic orbits of prescribed energy for autonomous Tonelli Lagrangian systems on the twisted cotangent bundle of a closed manifold.

CONTENTS

1. Introduction

The study of invariant sets plays a crucial role in the understanding of the properties of a dynamical system: it can be used to obtain information on the dynamics both at a local scale, such as the existence of nearby stable motions, and at a global one, such as the presence of chaos (see [Mos73]). In the realm of continuous flows periodic orbits are the simplest example of invariant sets and, therefore, they usually represent the first object of study. For systems admitting a Lagrangian formulation closed orbits received special consideration in the past years, in particular for the cases having geometrical or physical significance, such as geodesic flows [Kli78] or mechanical flows in phase space [Koz85]. In [Con06] Contreras formulated a very general theorem about the existence of periodic motions for autonomous Lagrangian systems over compact configuration spaces. This result was later analysed in detail by Abbondandolo, who discussed it in a series of lecture notes [Abb13]. It is the purpose of the present paper to give a generalization of such theorem to systems which admit only a *local* Lagrangian description (Theorem 1.6 below). Among these we find the important example of magnetic flows on surfaces, which we introduce in Section 1.6. We look at them in detail in the last part of this note: we will sketch a different method, devised by Taĭmanov in [Taĭ93], to find periodic orbits with low energy and we will study the *stability* of the energy levels, a purely symplectic property, which has important consequences for the existence of periodic orbits.

Let us start now our study by making precise the general setting in which we work.

1.1. **Twisted Lagrangian flows over closed manifolds.** Let M be a closed connected n-dimensional manifold and denote by

$$\pi : TM \ \longrightarrow \ M \qquad\qquad \pi : T^*M \ \longrightarrow \ M$$
$$(q,v) \ \longmapsto \ q \qquad\qquad\quad (q,p) \ \longmapsto \ q$$

the tangent and the cotangent bundle projection of M. Let us fix also an auxiliary Riemannian metric g on M and let $|\cdot|$ denote the associated norm.

Let $\sigma \in \Omega^2(M)$ be a closed 2-form on M which we refer to as the *magnetic form*. We call *twisted cotangent bundle* the symplectic manifold (T^*M, ω_σ), where $\omega_\sigma := d\lambda - \pi^*\sigma$. Here λ is the canonical 1-form defined by

$$\lambda_{(q,p)} \ = \ p \circ d_{(q,p)}\pi \,, \qquad \forall\, (q,p) \in T^*M \,.$$

If $K : T^*M \to \mathbb{R}$ is a smooth function, we denote by $t \mapsto \Phi_t^{(K,\sigma)}$ the Hamiltonian flow of K. It is generated by the vector field $X_{(K,\sigma)}$ defined by

$$\omega_\sigma(X_{(K,\sigma)}, \cdot) = -dK .$$

In local coordinates on T^*M such flow is obtained by integrating the equations

(1)
$$\begin{cases} \dot{q} = \dfrac{\partial K}{\partial p} , \\[2mm] \dot{p} = -\dfrac{\partial K}{\partial q} + \sigma\left(\dfrac{\partial K}{\partial p}, \cdot\right) . \end{cases}$$

The function K is an integral of motion for $\Phi^{(K,\sigma)}$. Moreover, if k is a regular value for K, then the flow lines lying on $\{K = k\}$ are tangent to the 1-dimensional distribution $\ker \omega_\sigma|_{\{K=k\}}$. This means that if $K' : T^*M \to \mathbb{R}$ is another Hamiltonian with a regular value k' such that $\{K' = k'\} = \{K = k\}$, then $\Phi^{(K',\sigma)}$ and $\Phi^{(K,\sigma)}$ are the same up to a *time reparametrization* on the common hypersurface. In other words, there exists a smooth family of diffeomorphisms $\tau_z : \mathbb{R} \to \mathbb{R}$ parametrized by $z \in \{K' = k'\} = \{K = k\}$ such that

$$\tau_z(0) = 0 \quad \text{and} \quad \Phi_t^{(K,\sigma)}(z) = \Phi_{\tau_z(t)}^{(K',\sigma)}(z) .$$

Hence, there is a bijection between the closed orbits of the two flows on the hypersurface.

Let $L : TM \to \mathbb{R}$ be a *Tonelli Lagrangian*. This means that for every $q \in M$, the restriction $L|_{T_qM}$ is superlinear and strictly convex (see [Abb13]):

(2)
$$\lim_{|v|\to+\infty} \frac{L(q,v)}{|v|} = +\infty , \quad \forall\, q \in M ,$$
$$\frac{\partial^2 L}{\partial v^2}(q,v) > 0 , \quad \forall\, (q,v) \in TM ,$$

where $\frac{\partial^2 L}{\partial v^2}(q,v)$ is the Hessian of $L|_{T_qM}$ at $v \in T_qM$. The *Legendre transform* associated to L is the fibrewise diffeomorphism

$$\mathcal{L} : TM \longrightarrow T^*M$$
$$(q,v) \longmapsto \frac{\partial L}{\partial v}(q,v) .$$

The *Legendre dual* of L is the *Tonelli Hamiltonian*

$$H : T^*M \longrightarrow \mathbb{R}$$
$$(q,p) \longmapsto p\big(\mathcal{L}^{-1}(q,p)\big) - L\big(\mathcal{L}^{-1}(q,p)\big) ,$$

which satisfies the analogue of (2) on T^*M. For every $k \in \mathbb{R}$, let $\Sigma_k^* := \{H = k\}$. These sets are compact and invariant for $\Phi^{(H,\sigma)}$. As a consequence such a flow is complete. We can use $\mathcal{L}$ to pull back to TM the Hamiltonian flow of H.

Definition 1.1. *Let $\Phi^{(L,\sigma)}$ be the flow on TM defined by conjugation*

$$\mathcal{L} \circ \Phi^{(L,\sigma)} = \Phi^{(H,\sigma)} \circ \mathcal{L} .$$

We call $\Phi^{(L,\sigma)}$ a twisted Lagrangian *flow and we write $X_{(L,\sigma)}$ for its generating vector field. Since $\Phi^{(H,\sigma)}$ is complete, $\Phi^{(L,\sigma)}$ is complete as well.*

The next proposition shows that the flow $\Phi^{(L,\sigma)}$ is locally a standard Lagrangian flow.

Proposition 1.2. *Let $U \subset M$ be an open set such that $\sigma|_U = d\theta$ for some $\theta \in \Omega^1(U)$. There holds*

$$X_{(L-\theta,0)} \; = \; X_{(L,\sigma)}|_U \, ,$$

where $L - \theta : TU \to \mathbb{R}$ is the Tonelli Lagrangian defined by $(L-\theta)(q,v) = L(q,v) - \theta_q(v)$ and $X_{(L-\theta,0)}$ is the standard Lagrange vector field of $L - \theta$.

The proof of this result follows from the next exercise.

Exercise 1. *Prove the following generalization of the Euler-Lagrange equations. Consider a smooth curve $\gamma : [0,T] \to M$. Then, the curve $(\gamma, \dot\gamma)$ is a flow line of $X_{(L,\sigma)}$ if and only if for every open set $W \subset M$ and every linear symmetric connection ∇ on W,*

$$(3) \qquad \left(\nabla_{\dot\gamma} \frac{\partial L}{\partial v} \right)(\gamma,\dot\gamma) \; = \; \frac{\partial L}{\partial q}(\gamma,\dot\gamma) \; + \; \sigma_\gamma(\dot\gamma, \cdot)$$

at every time $t \in [0,T]$ such that $\gamma(t) \in W$. In the above formula $\frac{\partial L}{\partial q} \in T^*M$ denotes the restriction of the differential of L to the horizontal distribution given by ∇.

1.2. The magnetic form.

Let $[\sigma] \in H^2(M;\mathbb{R})$ denote the cohomology class of σ. We observe that for any $\theta \in \Omega^1(M)$, there holds

$$X_{(L+\theta,\sigma+d\theta)} = X_{(L,\sigma)} \, .$$

Since $L + \theta$ is still a Tonelli Lagrangian, we expect that general properties of the dynamics depend on σ only via $[\sigma]$. Moreover, if $\theta \in \Omega^1(M)$ is defined by $\theta_q := -\frac{\partial L}{\partial v}(q,0)$, then

$$\min_{v \in T_q M} \left(L(q,v) \; + \; \theta_q(v) \right) \; = \; L(q,0) + \theta_q(0), \quad \forall q \in M \, .$$

Therefore, without loss of generality we assume from now on that $L|_{T_q M}$ attains its minimum at $(q,0)$, for every $q \in M$.

We can refine the classification of σ given by $[\sigma]$ by looking at the cohomological properties of its lift to the universal cover. Let $\widetilde\sigma$ be the pull-back of σ to the universal cover $\widetilde M \to M$. We say that σ is *weakly exact* if $[\widetilde\sigma] = 0$. This is equivalent to asking that

$$\int_{S^2} u^*\sigma \; = \; 0, \quad \forall \, u : S^2 \longrightarrow M \, .$$

We say that σ admits a *bounded weak primitive* if there is $\widetilde\theta \in \Omega^1(\widetilde M)$ such that $d\widetilde\theta = \widetilde\sigma$ and

$$\sup_{\widetilde q \in \widetilde M} |\widetilde\theta_{\widetilde q}| \; < \; +\infty \, .$$

In this case we write $[\widetilde\sigma]_b = 0$. Notice that both notions that we just introduced depend on σ only via $[\sigma]$.

Exercise 2. *If M is a surface and $[\sigma] \neq 0$, show that*
- *if $M = S^2$, then $[\widetilde\sigma] \neq 0$;*

- if $M = \mathbb{T}^2$, then $[\widetilde{\sigma}] = 0$, but $[\widetilde{\sigma}]_b \neq 0$;
- if $M \notin \{S^2, \mathbb{T}^2\}$, then $[\widetilde{\sigma}]_b = 0$.

Using the second point, prove that

- if $M = \mathbb{T}^n$ and $[\sigma] \neq 0$, then $[\widetilde{\sigma}] = 0$, but $[\widetilde{\sigma}]_b \neq 0$;
- if M is any manifold and $[\widetilde{\sigma}]_b = 0$, then

$$
\int_{\mathbb{T}^2} u^* \sigma \;=\; 0\,, \quad \forall\, u : \mathbb{T}^2 \longrightarrow M\,.
$$

1.3. **Energy.** As twisted Lagrangian flows are described by an autonomous Hamiltonian on the twisted cotangent bundle, they possess a natural first integral. It is the Tonelli function $E : TM \to \mathbb{R}$ given by $E := H \circ \mathcal{L}$. We call it the *energy* of the system and we write $\Sigma_k := \{E = k\}$, for every $k \in \mathbb{R}$. Let $V : M \to \mathbb{R}$ denote the restriction of E to the zero section and let $e_m(L)$ and $e_0(L)$ denote the minimum and maximum of V, respectively.

Proposition 1.3. *The energy can be written as*

$$
E(q,v) \;=\; \frac{\partial L}{\partial v}(q,v)(v) \;-\; L(q,v)
$$

and, for every $q \in M$, we have

$$
\min_{v \in T_q M} E(q,v) \;=\; E(q,0) \;=\; V(q) \;=\; -L(q,0)\,.
$$

Moreover,

- $k > e_0(L)$ *if and only if* $\pi : \Sigma_k \to M$ *is an S^{n-1}-bundle (isomorphic to the unit tangent bundle of M).*
- $k < e_m(L)$ *if and only if* $\Sigma_k = \emptyset$.

Exercise 3. *If $q_0 \in M$ is a critical point of V, then $(q_0, 0)$ is a constant periodic orbit of $\Phi^{(L,\sigma)}$ with energy $V(q_0)$.*

1.4. **The Mañé critical value of the universal cover.** When σ is weakly exact we define the *Mañé critical value* of the universal cover as

$$
\tag{4} c(L,\sigma) \;:=\; \inf_{d\widetilde{\theta} = \widetilde{\sigma}} \left(\sup_{\widetilde{q} \in \widetilde{M}} \widetilde{H}(\widetilde{q}, \widetilde{\theta}_{\widetilde{q}}) \right) \;\in\; \mathbb{R} \cup \{+\infty\}\,,
$$

where $\widetilde{H} : T^* \widetilde{M} \to \mathbb{R}$ is the lift of H to $\widetilde{M}$. This number plays an important role, since as it will be apparent from Theorem 1.6 and the examples in Section 1.6 the dynamics on Σ_k changes dramatically when k crosses $c(L,\sigma)$.

Proposition 1.4. *If σ is weakly exact, then*

- $c(L,\sigma) < +\infty$ *if and only if* $[\widetilde{\sigma}]_b = 0$;
- $c(L,\sigma) \geq e_0(L)$;
- *if $\sigma = d\theta_0$, where $\theta_0(\cdot) = \mathcal{L}(\cdot, 0)$, then $c(L,\sigma) = e_0(L)$ and the converse is true, provided $e_0(L) = e_m(L)$.*
- *given two Tonelli Lagrangians L_1 and L_2 and two real numbers k_1 and k_2 such that $\{H_1 = k_1\} = \{H_2 = k_2\}$, then*

$$
c(L_1,\sigma) \geq k_1 \iff c(L_2,\sigma) \geq k_2 \quad \text{and} \quad c(L_1,\sigma) \leq k_1 \iff c(L_2,\sigma) \leq k_2\,.
$$

1.5. **Example I: electromagnetic Lagrangians.** Let g be a Riemannian metric on M and $V : M \to \mathbb{R}$ be a function. Suppose that the Lagrangian is of *mechanical type*, namely it has the form

$$L(q, v) \ = \ \frac{1}{2}|v|^2 \ - \ V(q),$$

where $|\cdot|$ is the norm associated to g. In this case we refer to $\Phi^{(L,\sigma)}$ as a *magnetic flow* since we have the following physical interpretation of this system: it models the motion of a charged particle γ moving in M under the influence of a potential V and a stationary magnetic field σ. Using Exercise 1, the equation of motion reads

(5) $$\nabla_{\dot\gamma}\dot\gamma \ = \ -\nabla V(\gamma) \ + \ Y_\gamma(\dot\gamma),$$

where ∇V is the gradient of V and, for every $q \in M$, $Y_q : T_q M \to T_q M$ is defined by

$$g_q(Y_q(v_1), v_2) \ = \ \sigma_q(v_1, v_2), \quad \forall\, v_1, v_2 \in T_q M.$$

Exercise 4. *Prove that, if $k > \max V$, $\Phi^{(L,\sigma)}|_{\Sigma_k}$ can be described in terms of a purely kinetic system. Namely, define the Jacobi metric $g_k := \frac{k-V}{k}g$ and the Lagrangian $L_k(q,v) := \frac{1}{2}|v|_k^2$, where $|\cdot|_k$ is the norm induced by g_k. Using the Hamiltonian formulation, show that $\Phi^{(L,\sigma)}|_{\{E=k\}}$ is conjugated (up to time reparametrization) to $\Phi^{(L_k,\sigma)}|_{\{E_k=k\}}$, where E_k is the energy function of L_k.*

In the particular case $M = S^2$, magnetic flows describe yet another interesting mechanical system. Consider a rigid body in $\mathbb{R}^3$ with a fixed point and moving under the influence of a potential V. Suppose that V is invariant under rotations around the axis $\hat{z}$. We identify the rigid body as an element $\psi \in SO(3)$. Since $SO(3)$ is a Lie group, we use left multiplications to get $TSO(3) \simeq SO(3) \times \mathbb{R}^3 \ni (\psi, \Omega)$, where Ω is the angular speed of the body. Thus, we have a Lagrangian system on $SO(3)$ with $L = \frac{1}{2}|\Omega|^2 - V(\psi)$ and $\sigma = 0$. Here $|\cdot|$ denote the metric induced by the tensor of inertia of the body.

The quotient of $SO(3)$ by the action of the group of rotations around $\hat{z}$ is a two-sphere. The quotient map $q : SO(3) \to S^2$ sends ψ to the unit vector in $\mathbb{R}^3$, whose entries are the coordinates of $\hat{z}$ in the basis determined by ψ.

By the rotational symmetry, the quantity $\Omega \cdot \hat{z}$ is an integral of motion. Hence, for every $\omega \in \mathbb{R}$, the set $\{\Omega \cdot \hat{z} = \omega\} \subset TSO(3)$ is invariant under the flow and we have the commutative diagram

$$
\begin{array}{ccc}
\left(\{\Omega \cdot \hat{z} = \omega\}, X_{(L,0)}\right) & \xrightarrow{\ dq\ } & \left(TS^2, X_{(L_\omega, \sigma_\omega)}\right) \\
\ \ \downarrow{\scriptstyle \pi} & & \ \ \downarrow{\scriptstyle \pi} \\
SO(3) & \xrightarrow{\qquad q \qquad} & S^2.
\end{array}
$$

The resulting twisted Lagrangian system $(L_\omega, \sigma_\omega)$ on S^2 can be described as follows:

- $L_\omega(q, v) = \frac{1}{2}|v|^2 - V_\omega(q)$, where $|\cdot|$ is the norm associated to a *convex* metric g on S^2 (independent of ω) and V_ω is a potential (depending on ω);
- $\sigma_\omega = \omega \cdot \kappa$, where κ is the curvature form of g (in particular σ_ω has integral $4\pi\omega$ and, if $\omega \neq 0$, it is a symplectic form on S^2).

The rigid body model presented in this subsection is described in detail in [Kha79]. We refer the reader to [Nov82], for other relevant problems in classical mechanics that can be described in terms of twisted Lagrangian systems.

1.6. **Example II: magnetic flows on surfaces.** We now specialize further the example of electromagnetic Lagrangians that we discussed in the previous subsection and we consider purely kinetic systems on a closed oriented Riemannian surface (M, g). In this case

$$(6) \qquad L(q, v) := \frac{1}{2}|v|^2,$$

and $\sigma = f \cdot \mu$, where μ is the metric area form and $f : M \to \mathbb{R}$. The magnetic endomorphism can be written as $Y = f \cdot \imath$, where $\imath : TM \to TM$ is the fibrewise rotation by $\pi/2$.

Remark 1.5. *If the surface is isometrically embedded in the Euclidean space $\mathbb{R}^3$, Y is the classical Lorentz force. Namely, we have $Y_q(v) = v \times B(q)$, where $\times$ is the outer product of vectors in $\mathbb{R}^3$ and B is the vector field $B : M \to \mathbb{R}^3$ perpendicular to M and determined by the equation $\mathrm{vol}_{\mathbb{R}^3}(B, \cdot, \cdot) = \sigma$, where $\mathrm{vol}_{\mathbb{R}^3}$ is the Euclidean volume.*

For purely kinetic systems $E = L$ and, therefore, the solutions of the twisted Euler-Lagrange equations are parametrized by a multiple of the arc length. More precisely, if $(\gamma, \dot{\gamma}) \subset \Sigma_k$, then $|\dot{\gamma}| = \sqrt{2k}$. In particular, the solutions with $k = 0$ are exactly the constant curves. To characterise the solutions with $k > 0$ we write down explicitly the twisted Euler-Lagrange equation (5):

$$(7) \qquad \nabla_{\dot{\gamma}}\dot{\gamma} = f(\gamma) \cdot \imath\dot{\gamma}.$$

We see that γ satisfies (7) if and only if $|\dot{\gamma}| = \sqrt{2k}$ and

$$(8) \qquad \kappa_\gamma = s \cdot f(\gamma), \qquad s := \frac{1}{\sqrt{2k}},$$

where κ_γ is the geodesic curvature of γ. The advantage of working with Equation (8) is that it is invariant under orientation-preserving reparametrizations.

Let us do some explicit computations when the data are homogeneous. Thus, let g be a metric of constant curvature on M and let $\sigma = \mu$. When $M \neq \mathbb{T}^2$ we assume, furthermore, that the absolute value of the Gaussian curvature is 1. By (8), in order to find the trajectories of $\Phi^{(L,\sigma)}$ we need to solve the equation $\kappa_\gamma = s$ for all $s > 0$.

Denote by $\widetilde{M}$ the universal cover of M. Then, $\widetilde{S^2} = S^2$, $\widetilde{\mathbb{T}^2} = \mathbb{R}^2$ and, if M has genus larger than one, $\widetilde{M} = \mathbb{H}$, where $\mathbb{H}$ is the hyperbolic plane. Our strategy will be to study the trajectories of the lifted flow and then project them down to M. Working on the universal cover is easier since there the problem has a bigger symmetry group. Notice, indeed, that the lifted flow is invariant under the group of orientation preserving isometries $\mathrm{Iso}_+(\widetilde{M})$.

1.6.1. *The two-sphere.* Let us fix geodesic polar coordinates $(r, \varphi) \in (0, \pi) \times \mathbb{R}/2\pi\mathbb{Z}$ around a point $q \in S^2$ corresponding to $r = 0$. The metric takes the form $dr^2 + (\sin r)^2 d\varphi^2$. Let $C_r(q)$ be the boundary of the geodesic ball of radius r oriented in the counter-clockwise sense. We compute $\kappa_{C_r(q)} = \frac{1}{\tan r}$. Observe that $\tan r$ takes every positive value exactly once for $r \in (0, \pi/2)$. Therefore, if $s > 0$, the trajectories of the flow are all supported on $C_{r(s)}(q)$, where q varies in S^2 and

$$(9) \qquad r(s) = \arctan \frac{1}{s} \in (0, \pi/2).$$

In particular, all orbits are closed and their period is

$$T(s) = \frac{2\pi s}{\sqrt{s^2 + 1}} .$$

1.6.2. *The two-torus.* In this case we readily see that the trajectories of the lifted flow are circles of radius $r(s) = 1/s$. In particular, all the orbits are closed and contractible. Their period is $T(s) = 2\pi$, hence it is independent of s (or k).

1.6.3. *The hyperbolic surface.* We fix geodesic polar coordinates $(r, \varphi) \in (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ around a point $q \in \mathbb{H}$ corresponding to $r = 0$. The metric takes the form $dr^2 + (\sinh r)^2 d\varphi^2$. Defining $C_r(q)$ as in the case of S^2, we find $\kappa_{C_r(q)} = \frac{1}{\tanh r}$. Observe that $\tanh r$ takes all the values in $(0, 1)$ exactly once, for $r \in (0, +\infty)$. Therefore, if $s \in (1, +\infty)$, the trajectories of the flow are the closed curves $C_{r(s)}(q)$, where q varies in $\mathbb{H}$ and

(10) $$r(s) = \operatorname{arctanh} \frac{1}{s} \in (0, +\infty)$$

In particular, for s in this range all periodic orbits are contractible. The formula for the periods now reads

$$T(s) = \frac{2\pi s}{\sqrt{s^2 - 1}} .$$

To understand what happens, when $s \leq 1$ we take the upper half-plane as a model for the hyperbolic plane. Thus, let $\mathbb{H} = \{ z = (x, y) \in \mathbb{C} \mid y > 0 \}$. In these coordinates, the hyperbolic metric has the form $\frac{dx^2 + dy^2}{y^2}$ and

$$\operatorname{Iso}_+(\mathbb{H}) = \left\{ z \mapsto \frac{az + b}{cz + d} \ \middle| \ a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\} .$$

We readily see that the affine transformations $z \mapsto az$, with $a > 0$ form a subgroup of $\operatorname{Iso}_+(\mathbb{H})$. This subgroup preserves all the Euclidean rays from the origin and acts transitively on each of them. Hence, we conclude that such curves have constant geodesic curvatures. If $\varphi \in (0, \pi)$ is the angle made by such ray with the x-axis, we find that the geodesic curvature of such ray is $\cos \varphi$. In order to do such computation one has to write the metric using Euclidean polar coordinates centered at the origin. Using the whole isometry group, we see that all the segments of circle intersecting $\partial \mathbb{H}$ with angle φ have geodesic curvature $\cos \varphi$.

We claim that if $s \in (0, 1)$ and $\nu \neq 0$ is a free homotopy class of loops of M, there is a unique closed curve $\gamma_{s,\nu}$ in the class ν, which has geodesic curvature s. The class ν correspond to a conjugacy class in $\pi_1(M)$. We identify $\pi_1(M)$ with the set of deck transformations and we let $F : \mathbb{H} \to \mathbb{H}$ be a deck transformation belonging to the given conjugacy class. By a standard result in hyperbolic geometry, F has two fixed points on $\partial \mathbb{H}$ (remember, for example, that there exists a geodesic in $\mathbb{H}$ invariant under F). Then, $\gamma_{s,\nu}$ is the projection to M of the unique segment of circle connecting the fixed points of F and making an angle $\varphi = \arccos s$ with $\partial \mathbb{H}$. The uniqueness of $\gamma_{s,\nu}$ stems form the uniqueness of such segment of circle.

In a similar fashion, we consider the subgroup of $\operatorname{Iso}_+(\mathbb{H})$ made by the maps $z \mapsto z + b$, with $b \in \mathbb{R}$. It preserves the horizontal line $\{y = 1\}$ and act transitively on it. Hence, such curve has constant geodesic curvature. A computation shows that it is equal to 1, if it is oriented by ∂_x. Using the whole isometry group, we see that all the circles tangent to $\partial \mathbb{H}$ have geodesic curvature equal to 1. Following [Gin96] we see that there is no closed curve in M with such geodesic curvature. By

contradiction, if such curve exist, then its lift would be preserved by a non-constant deck transformation. We can assume without loss of generality that such lift is the line $\{y = 1\}$. We readily see that the only elements in $\text{Iso}_+(\mathbb{H})$ which preserve $\{y = 1\}$ are the horizontal translation. However, no such transformation can be a deck transformation, since it has only one fixed point on $\partial\mathbb{H}$.

Exercise 5. *Show that in this case* $c(L, \sigma) = \frac{1}{2}$.

1.7. The Main Theorem.
We are now ready to state the central result of this mini-course.

Theorem 1.6. *The following four statements hold.*

(1) *Suppose* $[\widetilde{\sigma}]_b = 0$. *For every* $k > c(L, \sigma)$,
  (a) *there exists a closed orbit on* Σ_k *in any non-trivial free homotopy class;*
  (b) *if* $\pi_{d+1}(M) \neq 0$ *for some* $d \geq 1$, *there exists a contractible orbit on* Σ_k.

(2) *Suppose* $[\widetilde{\sigma}] = 0$. *There exists a contractible orbit on* Σ_k, *for almost every energy* $k \in (e_0(L), c(L, \sigma))$.

(3) *Suppose* $[\widetilde{\sigma}] \neq 0$. *There exists a contractible orbit on* Σ_k, *for almost every energy* $k \in (e_0(L), +\infty)$.

(4) *There exists a contractible orbit on* Σ_k, *for almost every* $k \in (e_m(L), e_0(L))$.

The set for which existence holds in (2), (3) and (4) contains all the k's for which Σ_k^* is a stable hypersurface in (T^*M, ω_σ) (see [HZ94, page 122]).

In these notes, we will prove (1), (2) and (3) above by relating closed orbits of the flow to the zeros of a closed 1-form η_k on the space of loops on M. We introduce such form and prove some of its general properties in Section 2. In Section 3 we describe an abstract minimax method that we apply in Section 4 to obtain zeros of η_k in the specific cases listed in the theorem. A proof of (4) relies on different methods and it can be found in [AB14].

Remark 1.7. *When $[\sigma] = 0$, the theorem was proven by Contreras [Con06]. Point (1) and (2), with the additional hypothesis $[\widetilde{\sigma}]_b = 0$, were proven by Osuna [Osu05]. Point (2) was proven in [Mer10], for electromagnetic Lagrangians, and in [AB14] for general systems. A sketch of the proof of point (3) was given in [Nov82, Section 3] and in [Koz85, Section 3.2]. It was rigorously established in [AB14]. Point (4) follows by employing tools in symplectic geometry. For the weakly exact case it can also be proven using a variational approach as shown in [Abb13, Section 7]. For Lagrangians of mechanical type and vanishing magnetic form the existence problem in such interval has historically received much attention (see [Koz85, Section 2] and references therein).*

We end up this introduction by defining the notion of stability mentioned in the theorem.

1.8. Stable hypersurfaces.
In general, the dynamics on Σ_k^* may exhibit very different behaviours as k changes. However, given a regular energy level $\Sigma_{k_0}^*$, in some special cases we can find a new Hamiltonian $H' : T^*M \to \mathbb{R}$ such that $\{H' = k_0'\} = \Sigma_{k_0}^*$ and such that $\Phi^{(H',\sigma)}|_{\{H'=k_0'\}}$ and $\Phi^{(H',\sigma)}|_{\{H'=k'\}}$ are conjugated, up to a time reparametrization, provided k' is sufficiently close to k_0'.

Definition 1.8. *We say that an embedded hypersurface $\imath : \Sigma^* \longrightarrow T^*M$ is* stable *in the symplectic manifold (T^*M, ω_σ) if there exists an open neighbourhood W of Σ^* and a diffeomorphism $\Psi_W : \Sigma^* \times (-\varepsilon_0, \varepsilon_0) \to W$ with the property that:*

- $\Psi_W|_{\Sigma^* \times \{0\}} = \imath$;
- *the function $H^W : W \to \mathbb{R}$ defined through the commutative diagram*

$$\Sigma^* \times (-\varepsilon_0, \varepsilon_0) \xrightarrow{\Psi_W} W \,,$$

$$\text{pr}_2 \downarrow \quad \swarrow{}_{H^W}$$

$$(-\varepsilon_0, \varepsilon_0)$$

is such that, for every $k \in (-\varepsilon_0, \varepsilon_0)$,

$$\Phi^{(H^W, \sigma)}|_{\{H^W = 0\}} \quad and \quad \Phi^{(H^W, \sigma)}|_{\{H^W = k\}}$$

are conjugated by the diffeomorphism

$$w \mapsto \Psi_W(\imath^{-1}(w), k)$$

up to time reparametrization. In this case, the reparametrizing maps $\tau_{(z,k)}$ vary smoothly with $(z, k) \in \Sigma^* \times (-\varepsilon_0, \varepsilon_0)$ and satisfy $\tau_{(z,0)} = \mathrm{Id}_\mathbb{R}$, for all $z \in \Sigma^*$.

This implies that there is a bijection between the periodic orbits on $\Sigma^* = \{H^W = 0\}$ and those on $\{H^W = k\}$.

Thanks to a result of Macarini and G. Paternain [MP10], if Σ^* is the energy level of some Tonelli Hamiltonian, the function H^W can be taken to be Tonelli as well.

Proposition 1.9. *Suppose that for some $k > e_0(L)$, Σ_k^* is stable with stabilizing neighbourhood W. Up to shrinking W, there exists a Tonelli Hamiltonian $H_k : T^*M \to \mathbb{R}$ such that $H^W = H_k$ on W.*

In order to check whether an energy level is stable or not, we give the following necessary and sufficient criterion that can be found in [CM05, Lemma 2.3].

Proposition 1.10. *A hypersurface Σ_k^* is stable if and only if there exists $\alpha \in \Omega^1(\Sigma_k^*)$ such that*

$$(a) \ d\alpha(X_{(H,\sigma)}, \cdot) = 0, \qquad (b) \ \alpha(X_{(H,\sigma)})(z) \neq 0, \quad \forall z \in \Sigma_k^*.$$

In this case α is called a stabilizing *form. The first condition is implied by the following stronger assumption*

$$(a') \ d\alpha = \omega_\sigma|_{\Sigma_k^*} \,.$$

If (a') and (b) are satisfied we say that Σ_k^* is of contact type *and we call α a contact form. We distinguish between* positive *and* negative *contact forms according to the sign of the function $\alpha(X_{(H,\sigma)})$.*

In Section 6, we give some sufficient criteria for stability for magnetic flows on surfaces.

2. THE FREE PERIOD ACTION FORM

For the proof of the Main Theorem we need to characterize the periodic orbits on Σ_k via a variational principle on a space of loops. To this purpose we have first to adjust L.

2.1. **Adapting the Lagrangian.** Let us introduce a class of Tonelli Lagrangians whose fibrewise growth is quadratic. In this class we will be enabled to define the action functional on the space of loops with square-integrable velocity.

Definition 2.1. *We say that L is* quadratic at infinity *if there exists a metric g_∞ and a potential $V_\infty : M \to \mathbb{R}$ such that $L(q, v) = \frac{1}{2}|v|_\infty^2 - V_\infty(q)$ outside a compact set.*

The next result tells us that, if we look at the dynamics on a fixed energy level, it is not restrictive to assume that the Lagrangian is quadratic at infinity.

Proposition 2.2. *For any fixed $k \in \mathbb{R}$, there exists a Tonelli Lagrangian $L_k : TM \to \mathbb{R}$ which is quadratic at infinity and such that $L_k = L$ on $\{E \leq k_0\}$, for some $k_0 > k$. By choosing k_0 sufficiently large, we can obtain $e_0(L) = e_0(L_k)$ and, if $[\tilde{\sigma}] = 0$, also $c(L, \sigma) = c(L_k, \sigma)$.*

From now on, we assume that L is quadratic at infinity. In this case there exist positive constants C_0 and C_1 such that

$$(11) \qquad C_1|v|^2 - C_0 \leq L(q, v) \leq C_1|v|^2 + C_0, \quad \forall (q, v) \in TM.$$

An analogous statement holds for the energy.

2.2. **The space of loops.** We define the space of loops where the variational principle will be defined. Given $T > 0$, we call $W^{1,2}(\mathbb{R}/T\mathbb{Z}, M)$ the set

$$\left\{ \gamma : \mathbb{R}/T\mathbb{Z} \to M \;\middle|\; \gamma \text{ is absolutely continuous}, \int_0^T |\dot{\gamma}|^2 \, \mathrm{d}t < \infty \right\}.$$

Since we look for periodic orbits of arbitrary period, we want to let T vary among all the positive real numbers $\mathbb{R}^+$. This is the same as fixing the parametrization space to $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and keeping track of the period as an additional variable. Namely, we have the identification

$$\bigsqcup_{T>0} W^{1,2}(\mathbb{R}/T\mathbb{Z}, M) \longrightarrow \Lambda := W^{1,2}(\mathbb{T}, M) \times \mathbb{R}^+$$

$$\gamma(t) \longmapsto \big(x(s) := \gamma(sT), T\big).$$

Given a free homotopy class $\nu \in [\mathbb{T}, M]$, we denote by $W_\nu^{1,2} \subset W^{1,2}(\mathbb{T}, M)$ and $\Lambda_\nu \subset \Lambda$ the loops belonging to such class. We use the symbol 0 for the class of contractible loops.

Proposition 2.3. *The set Λ is a Hilbert manifold with $T_{(x,T)}\Lambda \simeq T_x W^{1,2} \times \mathbb{R}$, where $T_x W^{1,2} \simeq W^{1,2}(\mathbb{T}, x^*(TM))$ is the space of absolutely continuous vector fields along x with square-integrable covariant derivative. The metric on Λ is given by $g_\Lambda = g_{W^{1,2}} + \mathrm{d}T^2$, where*

$$(g_{W^{1,2}})_x(\xi_1, \xi_2) := \int_0^1 g_{x(s)}(\xi_1(s), \xi_2(s)) \, \mathrm{d}s + \int_0^1 g_{x(s)}(\xi_1'(s), \xi_2'(s)) \, \mathrm{d}s.$$

For any $T_- > 0$, $W^{1,2} \times [T_-, +\infty) \subset \Lambda$ is a complete metric space.

For more details on the space of loops we refer to [Abb13, Section 2] and [Kli78]. We end this subsection with two more definitions, which will be useful later on. First, we let

$$\frac{\partial}{\partial T} \ \in \ \Gamma(\Lambda)$$

denote the coordinate vector associated with the variable T. Then, if $x \in W^{1,2}$, we let

$$e(x) \ := \ \int_0^1 |x'|^2 \mathrm{d}s \qquad \text{and} \qquad \ell(x) \ := \ \int_0^1 |x'| \, \mathrm{d}s$$

be the L^2-*energy* and the *length* of x, respectively. We define analogous quantities for $\gamma \in \Lambda$. We readily see that $\ell(x) = \ell(\gamma)$ and $e(x) = Te(\gamma)$. Moreover, $\ell(x)^2 \leq e(x)$ holds.

2.3. The action form.

In this subsection, for every $k \in \mathbb{R}$, we construct $\eta_k \in \Omega^1(\Lambda)$, which vanishes exactly at the set of periodic orbits on Σ_k. Such 1-form will be made of two pieces: one depending only on L and k and one depending only on σ. The first piece will be the differential of the function

$$A_k : \Lambda \longrightarrow \mathbb{R}$$

$$\gamma \longmapsto \int_0^T \left[L(\gamma, \dot\gamma) + k \right] \mathrm{d}t \ = \ T \cdot \int_0^1 \left[L\left(x, \frac{x'}{T} \right) + k \right] \mathrm{d}s \, .$$

Such function is well-defined since L is quadratic at infinity (see (11)). It was proven in [AS09] that A_k is a $C^{1,1}$ function (namely, A_k is differentiable and its differential is locally uniformly Lipschitz-continuous).

In order to define the part of η_k depending on σ, we first introduce a differential form $\tau^\sigma \in \Omega^1(W^{1,2})$ called the *transgression* of σ. It is given by

$$\tau_x^\sigma(\xi) \ := \ \int_0^1 \sigma_{x(s)}(\xi(s), x'(s)) \, \mathrm{d}s \, , \quad \forall \, (x, \xi) \in TW^{1,2} \, .$$

By writing τ^σ in local coordinates, it follows that it is locally uniformly Lipschitz. If $u : [0, 1] \to W^{1,2}$ is a path of class C^1, then

$$(12) \qquad\qquad \int_0^1 u^* \tau^\sigma \ = \ \int_{[0,1] \times \mathbb{T}} \hat u^* \sigma \, ,$$

where $\hat u : [0, 1] \times \mathbb{T} \to M$ is the cylinder given by $\hat u(r, t) = u(r)(t)$. If $u_a : \mathbb{T} \to W^{1,2}$ is a homotopy of closed paths with parameter $a \in [0, 1]$, then we get a corresponding homotopy of tori $\hat u_a$. Since σ is closed, the integral of $\hat u_a^* \sigma$ on $\mathbb{T}^2$ is independent of a. We conclude that the integral of τ^σ on u_a does not depend on a either. Namely, τ^σ is a *closed form*.

Definition 2.4. *The* free period action form *at energy k is $\eta_k \in \Omega^1(\Lambda)$ defined as*

$$(13) \qquad\qquad \eta_k \ := \ dA_k \ - \ \mathrm{pr}_{W^{1,2}}^* \tau^\sigma \, ,$$

where $\mathrm{pr}_{W^{1,2}}^* : \Lambda \to W^{1,2}$ is the natural projection $(x, T) \mapsto x$.

Proposition 2.5. *The free period action form is closed and its zeros correspond to the periodic orbits of $\Phi^{(L,\sigma)}$ on Σ_k.*

The correspondence with periodic orbits follows by computing η_k explicitly on $TW^{1,2} \times 0$ and on $\frac{\partial}{\partial T}$. If $\xi \in TW^{1,2}$, then

$$(14) \qquad (\eta_k)_\gamma(\xi, 0) = \int_0^T \left[\frac{\partial L}{\partial q}(\gamma, \dot\gamma) \cdot \xi_T + \frac{\partial L}{\partial v}(\gamma, \dot\gamma) \cdot \dot\xi_T + \sigma_\gamma(\dot\gamma, \xi_T) \right] dt ,$$

where ξ_T is the reparametrization of ξ on $\mathbb{R}/T\mathbb{Z}$. In the direction of the period we have

$$
\begin{aligned}
(\eta_k)_\gamma\left(\frac{\partial}{\partial T} \right) = d_\gamma A_k\left(\frac{\partial}{\partial T} \right) &= \int_0^1 L\left(x, \frac{x'}{T} \right) ds + \\
&\quad + k - T \cdot \int_0^1 \frac{\partial L}{\partial v}\left(x, \frac{x'}{T} \right) \cdot \frac{x'}{T^2}\, ds \\
&= k - \int_0^1 E\left(x, \frac{x'}{T} \right) ds \\
&= k - \frac{1}{T} \int_0^T E(\gamma, \dot\gamma)\, dt .
\end{aligned}
$$

(15)

2.4. Vanishing sequences.
Our strategy to prove existence of periodic orbits will be to construct zeros of η_k by approximation.

Definition 2.6. *Let $\nu \in [\mathbb{T}, M]$ be a free homotopy class. A sequence $(\gamma_m) \subset \Lambda_\nu$ is called a* vanishing sequence *(at level k), if*

$$\lim_{m\to\infty} |\eta_k|_{\gamma_m} = 0 .$$

A limit point of a vanishing sequence is a zero of η_k. Thus, the crucial question is: when does a vanishing sequence admit limit points? Clearly, if $T_m \to 0$ or $T_m \to +\infty$ the set of limit points is empty. We now see that the opposite implication also holds.

Lemma 2.7. *If (γ_m) is a vanishing sequence, there exists $C > 0$ such that*

$$(16) \qquad e(x_m) \leq C \cdot T_m^2 .$$

Proof. We compute

$$C_1 \cdot \frac{e(x_m)}{T_m^2} - C_0 \overset{(\star)}{\leq} \int_0^1 E\left(x_m, \frac{x'_m}{T_m} \right) ds = k - \eta_{\gamma_m}^k\left(\frac{\partial}{\partial T} \right) \overset{(\star\star)}{\leq} k + \sup_m |\eta_k|_{\gamma_m} .$$

where in $(\star)$ we used (11) applied to E, and in $(\star\star)$ we used that

$$\left| \frac{\partial}{\partial T} \right| = 1 .$$

The desired estimate follows by observing that, since the sequence $\left(|\eta_k|_{\gamma_m} \right) \subset [0, +\infty)$ is infinitesimal, it is also bounded from above. $\qquad \square$

Proposition 2.8. *If (γ_m) is a vanishing sequence and $0 < T_- \leq T_m \leq T_+ < +\infty$ for some T_- and T_+, then (γ_m) has a limit point.*

Proof. By compactness of $[T_-, T_+]$, up to subsequences, $T_m \to T_\infty > 0$. By (16), the L^2-energy of x_m is uniformly bounded. Thus, (x_m) is uniformly 1/2-Hölder continuous. By the Arzelà-Ascoli theorem, up to subsequences, (x_m) converges uniformly to a continuous $x_\infty : \mathbb{T} \to M$. Therefore, x_m eventually belongs to a local chart $\mathcal{U}$ of $W^{1,2}$. In $\mathcal{U}$, η_k can be written as the differential of a standard action functional depending on time (see [AB14]) and the same argument contained in [Abb13, Lemma 5.3] when $\sigma = 0$ implies that (γ_m) has a limit point. $\square$

In order to construct vanishing sequences we will exploit some geometric properties of η_k. One of the main ingredients to achieve this goal will be to define a vector field on Λ generalizing the negative gradient vector field of the function A_k. We introduce it in the next subsection.

2.5. **The flow of steepest descent.** Let X_k denote the vector field on Λ defined by

$$X_k := -\frac{\sharp \eta_k}{\sqrt{1 + |\eta_k|^2}}$$

where $\sharp$ denote the duality between 1-forms and vector fields induced by g_Λ. Since X_k is locally uniformly Lipschitz, it gives rise to a flow which we denote by $r \mapsto \Phi_r^k$. For every $\gamma \in \Lambda$, we denote by $u_\gamma : [0, R_\gamma) \to \Lambda$ the maximal positive flow line starting at γ. We say that Φ^k is *positively complete* on a subset $Y \subset \Lambda$ if, for all $\gamma \in \Lambda$, either $R_\gamma = +\infty$ or there exists $R_{\gamma,Y} \in [0, R_\gamma)$ such that $u_\gamma(R_{\gamma,Y}) \notin Y$.

Except for the scaling factor $1/\sqrt{1 + |\eta_k|^2}$, the vector field X_k is the natural generalization of $-\nabla A_k = -\sharp(dA_k)$ to the case of non-vanishing magnetic form. We introduce such scaling so that $|X_k| \leq 1$ and we can give the following characterization of the flow lines u_γ with $R_\gamma < +\infty$.

Proposition 2.9. *Let $u : [0, R) \to \Lambda$ be a maximal positive flow line of X_k and for all $r \in [0, R)$ set $u(r) := \gamma(r) = (x(r), T(r))$. If $R < +\infty$, then there exists a sequence $(r_m)_{m \in \mathbb{N}} \subset [0, R)$ and a constant C such that*

$$(17) \quad \lim_{m \to \infty} r_m = R, \qquad \lim_{m \to \infty} T(r_m) = 0, \qquad e(x(r_m)) \leq C \cdot T(r_m)^2, \quad \forall m \in \mathbb{N}.$$

Proof. By contradiction, we suppose that $0 < T_- := \inf_{[0,R)} T(r)$. Since $|X_k| \leq 1$, u_γ is uniformly continuous and, by the completeness of $W^{1,2} \times [T_-, +\infty)$, there exists

$$\gamma_\infty := \lim_{r \to R} u(r).$$

By the existence theorem of solutions of ODE's, there exists a neighbourhood $\mathcal{B}$ of γ_∞ and $R_\mathcal{B} > 0$ such that

$$\forall \gamma \in \mathcal{B}, \quad r \longmapsto \Phi_r^k(\gamma) \text{ exists in } [0, R_\mathcal{B}].$$

This contradicts the fact that R is finite as soon as $r \in [0, R)$ is such that $\gamma(r) \in \mathcal{B}$ and $R - r < R_\mathcal{B}$. Therefore, $\inf T = 0$. Hence, we find a sequence $r_m \to R$ such that $T(r_m) \to 0$ and, for every m, $\frac{dT}{dr}(r_m) \leq 0$. The last property implies that

$$(18) \qquad 0 \geq \frac{dT}{dr}(r_m) = d_{u(r_m)}T(X_k) = -\frac{\eta_k\left(\frac{\partial}{\partial T}\right)}{\sqrt{1 + |\eta_k|^2}}(u(r_m)).$$

Finally, using Equation (15) and the estimates in (11), we have

$$
\begin{aligned}
0 \leq (\eta_k)_{u(r(m))} \left(\frac{\partial}{\partial T} \right) &= k - \int_0^1 E\left(x(r_m), \frac{x'(r_m)}{T(r_m)} \right) \mathrm{d}s \\
&\leq k - C_1 \int_0^1 \frac{|x'(r_m)|^2}{T(r_m)^2} \mathrm{d}s + C_0 \, .
\end{aligned}
$$

$\square$

The above proposition shows that flow lines whose interval of definition is finite come closer and closer to the subset of constant loops. As we saw in Lemma 2.7 the same is true for vanishing sequences with infinitesimal period. For these reasons in the next subsection we study the behaviour of the action form on the set of loops with short length.

2.6. **The subset of short loops.** We now define a local primitive for η_k close to the subset of constant loops. For $k > e_0(L)$, such primitive will enjoy some properties that will enable us to apply the minimax theorem of Section 3 to prove the Main Theorem. For our arguments we will need estimates which hold uniformly on a compact interval of energies. Hence, for the rest of this subsection we will suppose that a compact interval $I \subset (e_0(L), +\infty)$ is fixed.

Let $M_0 \subset W_0^{1,2}$ be the constant loops parametrized by $\mathbb{T}$ and $M_0 \times \mathbb{R}^+ \subset \Lambda_0$ the constant loops with arbitrary period. We readily see that $\tau^\sigma|_{M_0} = 0$. Thus, $\eta_k = dA_k|_{M_0 \times \mathbb{R}^+}$ and

$$
(19) \qquad A_k(x, T) = T(k - V(x)), \quad \forall (x, T) \in M_0 \times \mathbb{R}^+ \, .
$$

Now that we have described η_k on constant loops, let us see what happens nearby. First, we need the following lemma.

Lemma 2.10. *There exists $\delta_* > 0$ such that $\{\ell < \delta\} \subset W^{1,2}$ retracts with deformation on M_0, for all $\delta \leq \delta_*$. Thus, we have $\tau^\sigma|_{\{\ell < \delta_*\}} = dP^\sigma$, where*

$$
(20) \qquad
\begin{aligned}
P^\sigma : \{\ell < \delta_*\} &\longrightarrow \mathbb{R} \\
x &\longmapsto \int_{B^2} \hat{u}_x^* \sigma \, ,
\end{aligned}
$$

where $\hat{u}_x : B^2 \to M$ is the disc traced by x under the action of the deformation retraction. Furthermore, there exists $C > 0$ such that

$$
(21) \qquad |P^\sigma(x)| \leq C \cdot \ell(x)^2 \, .
$$

Proof. Choose $\delta < 2\rho(g)$, where $\rho(g)$ is the injectivity radius of g. With this choice, for each $x \in \{\ell < \delta\}$ and each $s \in \mathbb{T}$, there exists a unique geodesic $y_s : [0, 1] \to M$ joining $x(0)$ to $x(s)$. For each $a \in [0, 1]$ define $x_a : \mathbb{T} \to M$ by $x_a(s) := y_s(a)$. Taking a smaller δ if necessary, one can prove that $a \mapsto |x_a'|$ is a non-decreasing family of functions (use normal coordinates at $x(0)$). Thus, $a \mapsto \ell(x_a)$ is non-decreasing as well and

$$
\begin{aligned}
[0, 1] \times \{\ell < \delta\} &\longrightarrow \{\ell < \delta\} \\
(a, x) &\longmapsto x_a
\end{aligned}
$$

yields the desired deformation. In order to estimate P^σ is enough to bound the area of the deformation disc $\hat{u}_x$:

$$
\begin{aligned}
\text{area}(\hat{u}_x) &\leq \int_0^1 da \int_0^1 \left| \frac{dy_s}{da}(a) \right| \cdot |x_a'(s)| \, ds \\
&\leq \int_0^1 da \int_0^1 d(x(0), x(s)) |x'(s)| \, ds \leq \frac{\ell(x)}{2} \ell(x) \,.
\end{aligned}
$$

$\square$

In view of this lemma, for all $\delta \in (0, \delta_*]$, we define the set

$$
\text{(22)} \qquad \mathcal{V}^\delta := \{\ell < \delta\} \times \mathbb{R}^+ \subset \Lambda_0
$$

and the function $S_k : \mathcal{V}^{\delta_*} \longrightarrow \mathbb{R}$ given by

$$
\text{(23)} \qquad S_k := A_k - P^\sigma \circ \mathrm{pr}_{W^{1,2}} \,.
$$

Such a function is a primitive of η_k on $\mathcal{V}^{\delta_*}$. By (11), it admits the following upper bound.

Proposition 2.11. *There exists $C > 0$ such that, for every $\gamma \in \mathcal{V}^{\delta_*}$, there holds*

$$
\text{(24)} \qquad S_k(\gamma) \leq C \cdot \left(\frac{e(x)}{T} + T + \ell(x)^2 \right), \quad \forall k \in I \,.
$$

This result has an immediate consequence on vanishing sequences and flow lines of Φ^k.

Corollary 2.12. *Let $b > 0$ and $k \in I$ be fixed. The following two statements hold:*
(1) if (γ_m) is a vanishing sequence such that $\gamma_m \notin \{S_k < b\}$ for all $m \in \mathbb{N}$, then T_m is bounded away from zero;
(2) the flow Φ^k is positively complete on the set $\Lambda \setminus \{S_k < b\}$.

We conclude this section by showing that the infimum of S_k on short loops is zero and it is approximately achieved on constant loops with small period. Furthermore, S_k is bounded away from zero on the set of loops having some fixed positive length.

Proposition 2.13. *There exist $\delta_I \leq \delta_*$ and positive numbers b_I, T_I such that, for all $k \in I$,*

$$
\text{(25)} \quad (a) \ \inf_{\mathcal{V}^{\delta_I}} S_k = 0, \qquad (b) \ \inf_{\partial \mathcal{V}^{\delta_I}} S_k \geq b_I, \qquad (c) \ \sup_{M_0 \times \{T_I\}} S_k < \frac{b_I}{2} \,.
$$

Proof. Since for all $q \in M$, the function $L|_{T_q M}$ attains its minimum at $(q, 0)$, the estimate from below on L obtained in (11) can be refined to

$$
L(q, v) \geq C_1 |v|^2 + \min_{q \in M} L(q, 0) = C_1 |v|^2 - e_0(L) \,.
$$

From this inequality and (21), we can bound from below $S_k(\gamma)$:

$$
\begin{aligned}
S_k(\gamma) &\geq T \cdot \int_0^1 \left[C_1 \cdot \frac{|x'|^2}{T^2} - e_0(L) + k \right] ds - C \cdot \ell(x)^2 \\
&\geq C_1 \cdot \frac{e(x)}{T} + (k - e_0(L)) \cdot T - C \cdot \ell(x)^2 \\
&\overset{(\star)}{\geq} 2\sqrt{C_1(\min I - e_0(L))} \cdot \ell(x) - C \cdot \ell(x)^2 \,.
\end{aligned}
$$

where in $(\star)$ we made use of the inequality between arithmetic and geometric mean. Hence, there exists $\delta_I > 0$ sufficiently small, such that the last quantity is positive if $\ell(x) < \delta_I$ and bounded from below by

$$b_I := 2\sqrt{C_1(\min I - e_0(L))} \cdot \delta_I - C \cdot \delta_I^2 > 0$$

if $\ell(x) = \delta_I$. This implies Inequality *(b)* in (25) and that $\inf_{V^{\delta_I}} S_k \geq 0$. To prove that $\inf_{V^{\delta_I}} S_k \leq 0$ and that there exists T_I such that Inequality *(c)* in (25) holds, we just recall from (19) that

$$\lim_{T \to 0} \sup_{M_0 \times \{T\}} S_k = 0. \qquad \square$$

In the next section we will prove a minimax theorem for a class of closed 1-form on abstract Hilbert manifolds. Such a class will satisfy a general version of the properties we have proved so far for η_k.

3. The minimax technique

In this section we present an abstract minimax technique which represents the core of the proof of the Main Theorem. We formulate it in a very general form on a non-empty Hilbert manifold $\mathscr{H}$.

3.1. **An abstract theorem.** We start by setting some notation for homotopy classes of maps from Euclidean balls into $\mathscr{H}$. Let $d \in \mathbb{N}$ and $\mathscr{U}$ be a subset of $\mathscr{H}$. Define $\left[(B^d, \partial B^d), (\mathscr{H}, \mathscr{U})\right]$ as the set of homotopy classes of maps $\gamma : (B^d, \partial B^d) \to (\mathscr{H}, \mathscr{U})$. By this we mean that the maps send B^d to $\mathscr{H}$ and ∂B^d to $\mathscr{U}$, and that the homotopies do the same. The classes $[\gamma]$, where γ is such that $\gamma(B^d) \subset \mathscr{U}$ are called *trivial*. If $\mathscr{U}' \subset \mathscr{U}$, we have a map

$$i_{\mathscr{U}}^{\mathscr{U}'} : \left[(B^d, \partial B^d), (\mathscr{H}, \mathscr{U}')\right] \longrightarrow \left[(B^d, \partial B^d), (\mathscr{H}, \mathscr{U})\right]$$

We are now ready to state the main result of this section.

Theorem 3.1. *Let $\mathscr{H}$ be a non-empty Hilbert manifold, $\mathscr{I} = [k_0, k_1]$ be a compact interval and $d \geq 1$ an integer. Let $\alpha_k \in \Omega^1(\mathscr{H})$ be a family of Lipschitz-continuous forms parametrized by $k \in \mathscr{I}$ and such that*

- *the integral of α_k over contractible loops vanishes;*
- *$\alpha_k = \alpha_{k_0} + (k - k_0)d\mathscr{T}$, where $\mathscr{T} : \mathscr{H} \to (0, +\infty)$ is a $C^{1,1}$ function such that*

$$(26) \qquad \sup_{\mathscr{H}} |d\mathscr{T}| < +\infty.$$

Define the vector field

$$(27) \qquad \mathscr{X}_k := -\frac{\sharp \alpha_k}{\sqrt{1 + |\alpha_k|^2}},$$

where $\sharp$ is the metric duality, and suppose that there exists an open set $\mathscr{V} \subset \mathscr{H}$ such that:

- *there exists $\mathscr{S}_k : \overline{\mathscr{V}} \to \mathbb{R}$ satisfying*

$$(28) \qquad d\mathscr{S}_k = \alpha_k, \qquad \mathscr{S}_k = \mathscr{S}_{k_0} + (k - k_0)\mathscr{T};$$

- *there exists a real number*

(29) $$\beta_0 \; < \; \inf_{\partial \mathscr{V}} \mathscr{S}_{k_0} \; =: \; \beta_{\partial \mathscr{V}}$$

  such that the flow $r \mapsto \Phi_r^{\mathscr{X}_k}$ *is positively complete on the set* $\mathscr{H} \backslash \{\mathscr{S}_k < \beta_0\}$;
- *there exists a set* $\mathscr{M} \subset \{\mathscr{S}_{k_1} < \beta_0\}$ *and a class* $\mathscr{G} \in [(B^d, \partial B^d), (\mathscr{H}, \mathscr{M})]$
  such that $i_{\mathscr{V}}^{\mathscr{M}}(\mathscr{G})$ *is non-trivial.*

Then, the following two statements hold true. First, for all $k \in \mathscr{I}$, *there exists a sequence* $(h_m^k)_{m \in \mathbb{N}} \subset \mathscr{H} \setminus \{\mathscr{S}_k < \beta_0\}$ *such that*

$$\lim_{m \to \infty} |\alpha_k|_{h_m^k} \; = \; 0 \, .$$

Second, there exists a subset $\mathscr{I}_* \subset \mathscr{I}$ *such that*

- $\mathscr{I} \setminus \mathscr{I}_*$ *is negligible with respect to the 1-dimensional Lebesgue measure;*
- *for all* $k \in \mathscr{I}_*$ *we have*

$$\sup_{m \in \mathbb{N}} \mathscr{T}(h_m^k) \; < \; +\infty \, .$$

Moreover, if there exists a $C^{1,1}$-*function* $\widehat{\mathscr{S}_k} : \mathscr{H} \to \mathbb{R}$ *which extends* $\mathscr{S}_k$ *and satisfies* (28) *on the whole* $\mathscr{H}$, *we also have that*

(30) $$\lim_{m \to \infty} \widehat{\mathscr{S}_k}(h_m^k) \; = \; \inf_{\gamma \in \mathscr{G}} \sup_{\xi \in B^d} \widehat{\mathscr{S}_k} \circ \gamma \, (\xi) \; \geq \; \beta_{\partial \mathscr{V}} \, .$$

To prove Theorem 1.6*(1a)* we will also need a version of the minimax theorem for $d = 0$, namely when the maps are simply points in $\mathscr{H}$. We state it here for a single function and not for a 1-parameter family since this will be enough for the intended application. For a proof we refer to [Abb13, Remark 1.10].

Theorem 3.2. *Let* $\mathscr{H}$ *be a non-empty Hilbert manifold and let* $\widehat{\mathscr{S}} : \mathscr{H} \to \mathbb{R}$ *be a* $C^{1,1}$-*function bounded from below. Suppose that the flow of the vector field*

$$\mathscr{X} \; := \; - \frac{\nabla \widehat{\mathscr{S}}}{\sqrt{1 + |\nabla \widehat{\mathscr{S}}|^2}}$$

is positively complete on some non-empty sublevel set of $\widehat{\mathscr{S}}$. *Then, there exists a sequence* $(h_m)_{m \in \mathbb{N}} \subset \mathscr{H}$ *such that*

(31) $$\lim_{m \to +\infty} |d_{h_m} \widehat{\mathscr{S}}| \; = \; 0 \, , \qquad \lim_{m \to +\infty} \widehat{\mathscr{S}}(h_m) \; = \; \inf_{\mathscr{H}} \widehat{\mathscr{S}} \, .$$

In the next two subsections we prove Theorem 3.1. First, we introduce some preliminary definitions and lemmas and then we present the core of the argument.

3.2. **Preliminary results.** We start by defining the *variation* of the 1-form α_k along any path $u : [a_0, a_1] \to \mathscr{H}$. It is the real number

(32) $$\alpha_k(u) \; := \; \int_{a_0}^{a_1} \alpha_k \left(\frac{du}{da} \right) (u(a)) \, \mathrm{d}a \, .$$

We collect the properties of the variation along a path in a lemma.

Lemma 3.3. *If* u *is a path in* $\mathscr{H}$ *and* $\overline{u}$ *is the inverse path, we have*

(33) $$\alpha_k(\overline{u}) \; = \; - \alpha_k(u) \, .$$

If u_1 and u_2 are two paths in $\mathscr{H}$ such that the ending point of u_1 coincides with the starting point of u_2, we denote by $u_1 * u_2$ the concatenation of the two paths and we have

$$(34) \qquad \alpha_k(u_1 * u_2) \;=\; \alpha_k(u_1) \;+\; \alpha_k(u_2) \,,$$

If u is a contractible closed path in $\mathscr{H}$, we have

$$(35) \qquad \alpha_k(u) \;=\; 0 \,.$$

Finally, let $\gamma : Z \to \mathscr{H}$ be any smooth map from a Hilbert manifold Z such that there exists a function $\mathscr{S}_k^\gamma : Z \to \mathscr{H}$ with the property that

$$(36) \qquad d\mathscr{S}_k^\gamma \;=\; \gamma^* \alpha_k \,.$$

Then, for all paths $z : [a_0, a_1] \to Z$ we have

$$(37) \qquad \alpha_k(\gamma \circ z) \;=\; \mathscr{S}_k^\gamma(z(a_1)) \;-\; \mathscr{S}_k^\gamma(z(a_0)) \,.$$

Let us come back to the statement of Theorem 3.1. Fix a point $\xi_* \in \partial B^d$ and for every $\gamma \in \mathscr{G}$ define the unique $\mathscr{S}_k^\gamma : B^d \to \mathscr{H}$ such that

$$(38) \qquad d\mathscr{S}_k^\gamma \;=\; \gamma^* \alpha_k \,, \qquad \mathscr{S}_k^\gamma(\xi_*) \;=\; \mathscr{S}_k(\gamma(\xi_*)) \,.$$

We observe that this is a good definition since B^d is simply connected and $\gamma(\xi_*)$ belongs to the domain of definition of $\mathscr{S}_k$ as $\gamma \in \mathscr{G}$. Moreover, if α_k admits a global primitive $\widehat{\mathscr{S}_k}$ on $\mathscr{H}$ extending $\mathscr{S}_k$, then clearly we have $\mathscr{S}_k^\gamma = \widehat{\mathscr{S}_k} \circ \gamma$. Finally, thanks to the previous lemma, for every $\xi \in B^d$ we have the formula

$$(39) \qquad \mathscr{S}_k^\gamma(\xi) \;=\; \mathscr{S}_k(\gamma(\xi_*)) \;+\; \alpha_k(\gamma \circ z_\xi) \,,$$

where $z_\xi : [0, 1] \to B^d$ is any path connecting ξ_* and ξ.

Remark 3.4. *If $d \neq 1$, then $\mathscr{S}_k^\gamma$ does not depend on the choice of the point $\xi_* \in \partial B^d$ as $S^{d-1} = \partial B^d$ is connected. On the other hand, if $d = 1$ there are two possible choices for ξ_* and the two corresponding primitives of $\gamma^* \eta_k$ differ by a constant, which depends only on the class $\mathscr{G}$ and not on γ.*

Definition 3.5. *We define the* minimax function *$c_{\mathscr{G}} : \mathscr{I} \to \mathbb{R} \cup \{-\infty\}$ by*

$$(40) \qquad c_{\mathscr{G}}(k) := \inf_{\gamma \in \mathscr{G}} \; \sup_{\xi \in B^d} \; \mathscr{S}_k^\gamma(\xi) \,.$$

In the next lemma we show that $c_{\mathscr{G}}(k)$ is finite and that, for each $\gamma \in \mathscr{G}$, the points almost realizing the supremum of the function $\mathscr{S}_k^\gamma$ lie in the complement of the set $\{\mathscr{S}_k < \beta_0\}$.

Lemma 3.6. *Let $k \in \mathscr{I}$ and $\gamma \in \mathscr{G}$. There holds*

$$(41) \qquad \sup_{B^d} \; \mathscr{S}_k^\gamma \;\geq\; \beta_{\partial \mathscr{V}} \,.$$

Moreover, if $\beta_1 < \beta_{\partial \mathscr{V}}$, then $\forall \, \xi \in B^d$ the following implication holds

$$(42) \qquad \mathscr{S}_k^\gamma(\xi) \geq \sup_{B^d} \mathscr{S}_k^\gamma - (\beta_{\partial \mathscr{V}} - \beta_1) \quad \Longrightarrow \quad \gamma(\xi) \notin \{\mathscr{S}_k < \beta_1\} \,.$$

Proof. Since $i_{\mathscr{V}}^{\mathscr{M}}(\mathscr{G})$ is non-trivial, the set $\{\xi \in B^d \,|\, \gamma(\xi) \in \partial\mathscr{V}\}$ is non-empty. Therefore, there exists an element $\widehat{\xi}$ in this set and a path $z_{\widehat{\xi}} : [0,1] \to B^d$ from ξ_* to $\widehat{\xi}$ such that $\gamma \circ z_{\widehat{\xi}}|_{[0,1)} \subset \mathscr{V}$. By (39) and (37) we have

$$
\begin{aligned}
\mathscr{S}_k^\gamma(\widehat{\xi}) &= \mathscr{S}_k(\gamma(\xi_*)) + \alpha_k(\gamma \circ z_{\widehat{\xi}}) \\
&= \mathscr{S}_k(\gamma(\xi_*)) + \left(\mathscr{S}_k(\gamma(\widehat{\xi})) - \mathscr{S}_k(\gamma(\xi_*))\right) = \mathscr{S}_k(\gamma(\widehat{\xi})),
\end{aligned}
$$

which implies (41) by (29). In order to prove the second statement we consider $\xi \in B^d$ such that $\gamma(\xi) \in \{\mathscr{S}_k < \beta_1\}$. Without loss of generality there exists a path $z_{\xi,\widehat{\xi}} : [0,1] \to B^d$ from ξ to $\widehat{\xi}$ such that $z_{\xi,\widehat{\xi}}|_{[0,1)} \subset \mathscr{V}$. Using (37) twice, we compute

$$
\begin{aligned}
\sup_{B_d} \mathscr{S}_k^\gamma \geq \mathscr{S}_k^\gamma(\widehat{\xi}) &= \mathscr{S}_k^\gamma(\xi) + \alpha_k(\gamma \circ z_{\xi,\widehat{\xi}}) \\
&= \mathscr{S}_k^\gamma(\xi) + \left(\mathscr{S}_k(\gamma(\widehat{\xi})) - \mathscr{S}_k(\gamma(\xi))\right) > \mathscr{S}_k^\gamma(\xi) + \left(\beta_{\partial\mathscr{V}} - \beta_1\right),
\end{aligned}
$$

which yields the contrapositive of the implication we had to show. □

We now see that, since the family $k \mapsto \alpha_k$ is monotone in the parameter k, the same is true for the numbers $c_{\mathscr{G}}(k)$.

Lemma 3.7. *If $k_2 \leq k_3$ and $\gamma \in \mathscr{G}$, we have*

$$
(43) \qquad\qquad \mathscr{S}_{k_3}^\gamma = \mathscr{S}_{k_2}^\gamma + (k_3 - k_2)\,\mathscr{T} \circ \gamma.
$$

As a consequence, $c_{\mathscr{G}}$ is a non-decreasing function.

Proof. We observe that

- $d\left(\mathscr{S}_{k_3}^\gamma - \mathscr{S}_{k_2}^\gamma\right) = \gamma^*\left(\alpha_{k_3} - \alpha_{k_2}\right) = \gamma^*\left((k_3 - k_2)\,d\mathscr{T}\right)$
- $\mathscr{S}_{k_3}^\gamma(\xi_*) - \mathscr{S}_{k_2}^\gamma(\xi_*) = \mathscr{S}_{k_3}(\gamma(\xi_*)) - \mathscr{S}_{k_2}(\gamma(\xi_*)) = (k_3 - k_2)\,\mathscr{T}(\gamma(\xi_*))$.

These two equalities imply that the function $\mathscr{S}_{k_2}^\gamma + (k_3 - k_2)\mathscr{T} \circ \gamma$ satisfies (38) with $k = k_3$. Since these conditions identify a unique function, equation (43) follows. In particular, we have $\mathscr{S}_{k_2}^\gamma \leq \mathscr{S}_{k_3}^\gamma$. Taking the inf-sup of this inequality on $\mathscr{G}$, we get $c_{\mathscr{G}}(k_2) \leq c_{\mathscr{G}}(k_3)$. □

We end this subsection by adjusting the vector field $\mathscr{X}_k$ so that its flow becomes positively complete on all $\mathscr{H}$. We fix $\beta_1 \in (\beta_0, \beta_{\partial\mathscr{V}})$ and let $\mathscr{B} : [\beta_0, \beta_1] \to [0,1]$ be a function that is equal to 0 in a neighbourhood of β_0 and equal to 1 in a neighbourhood of β_1. We set

$$
\check{\mathscr{X}}_k := (\mathscr{B} \circ \mathscr{S}_k) \cdot \mathscr{X}_k \in \Gamma(\mathscr{H}).
$$

We observe that

- $\check{\mathscr{X}}_k = 0$ on $\{\mathscr{S}_k < \beta_0\}$,   • $\check{\mathscr{X}}_k = \mathscr{X}_k$ on $\mathscr{H} \setminus \{\mathscr{S}_k < \beta_1\}$,

and, hence, the flow $\Phi^{\check{\mathscr{X}}_k}$ is positively complete.

3.3. **Proof of Theorem 3.1.** Let us define $\mathscr{I}_*$ as the set

$$\left\{ k \in [k_0, k_1] \,\middle|\, \exists C(k_*) \text{ such that } c_{\mathscr{G}}(k) - c_{\mathscr{G}}(k_*) \le C(k_*)(k - k_*), \; \forall k \in [k_*, k_1] \right\}.$$

Namely, $\mathscr{I}_*$ is the set of points at which $c_{\mathscr{G}}$ is Lipschitz-continuous on the right. Since $c_{\mathscr{G}}$ is a non-decreasing real function, by Lebesgue Differentiation Theorem, $c_{\mathscr{G}}$ is Lipschitz-continuous at almost every point. In particular, $\mathscr{I} \setminus \mathscr{I}_*$ has measure zero.

We are now ready to show that

(1) for all $k \in \mathscr{I}$, there exists a vanishing sequence $(h_m^k)_{m \in \mathbb{N}} \subset \mathscr{H} \setminus \{\mathscr{S}_k < \beta_0\}$ and that

(2) for all $k_* \in \mathscr{I}_*$, such vanishing sequence can be taken to satisfy

$$\sup_{m \in \mathbb{N}} \mathscr{T}(h_m^{k_*}) \; < \; C(k_*) \; + \; 3\,.$$

We will prove only the statement about the vanishing sequences with parameter in $\mathscr{I}_*$, as the argument can be easily adapted to prove the statement for a general parameter in $\mathscr{I}$.

We assume by contradiction that there exists a positive number ε_0 such that

(44) $$|\alpha_{k_*}| \; \ge \; \varepsilon_0\,, \quad \text{on } \{\mathscr{T} < C(k_*) + 3\} \setminus \{\mathscr{S}_{k_*} < \beta_1\}\,.$$

Consider a decreasing sequence $(k_m)_{m \in \mathbb{N}} \subset (k_*, k_1]$ such that $k_m \to k_*$. Set $\delta_m := k_m - k_*$ and take a corresponding sequence $(\gamma_m)_{m \in \mathbb{N}} \subset \mathscr{G}$ such that

$$\sup_{B^d} \mathscr{S}_{k_m}^{\gamma_m} \; < \; c_{\mathscr{G}}(k_m) \; + \; \delta_m\,.$$

For every $\xi \in B^d$ we consider the sequence of flow lines

$$u_m^\xi : [0, 1] \longrightarrow \mathscr{H}$$

$$r \longmapsto \Phi_r^{\mathscr{X}_{k_*}}(\gamma_m(\xi))\,.$$

Conversely, for any time parameter $r \in [0, 1]$, we get the map

(45) $$\gamma_m^r \; := \; \Phi_r^{\mathscr{X}_{k_*}}(\gamma_m)\,.$$

We readily see that $\gamma_m^r|_{\partial B^d} = \gamma_m|_{\partial B^d}$ and $\gamma_m^r \in \mathscr{G}$. In particular, for every $\xi \in B^d$ and $r \in [0, 1]$ the concatenated curve

(46) $$\left(\gamma_m \circ z_\xi\right) \; * \; u_m^\xi|_{[0,r]} \; * \; \left(\overline{\gamma_m^r \circ z_\xi}\right)$$

is contractible. Therefore, Lemma 3.3 and Equation (39) yield

(47) $$\mathscr{S}_{k_*}^{\gamma_m^r}(\xi) \; = \; \mathscr{S}_{k_*}^{\gamma_m}(\xi) \; + \; \alpha_{k_*}(u_m^\xi|_{[0,r]})\,.$$

Finally, since u_m^ξ is a flow line, we have

(48) $$\alpha_{k_*}(u_m^\xi|_{[0,r]}) \; = \; \int_0^r \alpha_{k_*}\left(-\frac{\mathscr{B} \cdot \sharp \alpha_{k_*}}{\sqrt{1 + |\alpha_{k_*}|^2}}\right)(u_m^\xi(\rho))\, \mathrm{d}\rho$$

$$= \; -\int_0^r \frac{\mathscr{B} \cdot |\alpha_{k_*}|^2}{\sqrt{1 + |\alpha_{k_*}|^2}}(u_m^\xi(\rho))\, \mathrm{d}\rho\,.$$

Therefore $\alpha_{k_*}(u_m^\xi|_{[0,r]}) \le 0$ and we find that, for every $m \in \mathbb{N}$,

(49) $$r \longmapsto \mathscr{S}_{k_*}^{\gamma_m^r} \quad \text{is a non-increasing family of functions on } B^d\,.$$

Let us estimate the supremum of $\mathscr{S}_{k_*}^{\gamma_m^r}$. When $r = 0$, (43) and the definition of $\mathscr{I}_*$ imply:

$$(50) \qquad \sup_{B^d} \mathscr{S}_{k_*}^{\gamma_m} \;\leq\; \sup_{B^d} \mathscr{S}_{k_m}^{\gamma_m} \;<\; c_{\mathscr{G}}(k_m) + \delta_m \;\leq\; c_{\mathscr{G}}(k_*) + (C(k_*) + 1)\,\delta_m\,.$$

Thus, by (49) we get, for every $r \in [0,1]$,

$$(51) \qquad\qquad \sup_{B^d} \mathscr{S}_{k_*}^{\gamma_m^r} \;<\; c_{\mathscr{G}}(k_*) + (C(k_*) + 1)\,\delta_m\,.$$

If $r \in [0,1]$, we define the sequence of subsets of B^d

$$J_m^r \;:=\; \big\{\, \mathscr{S}_{k_*}^{\gamma_m^r} \,>\, c_{\mathscr{G}}(k_*) - \delta_m \,\big\}\,.$$

Let us give a closer look to these sets. First, we observe that if $\xi \in J_m^r$, then (47) and (51) imply that
$$(52)$$
$$\alpha_{k_*}(u_m^\xi|_{[0,r]}) \;>\; c_{\mathscr{G}}(k_*) - \delta_m \;-\; \big(c_{\mathscr{G}}(k_*) + (C(k_*) + 1)\,\delta_m \big) \;=\; -(C(k_*) + 2)\,\delta_m\,.$$

Then, we claim that for m large enough

$$(53) \quad \xi \in J_m^r \;\implies\; \gamma_m^r(\xi) \,\in\, \big\{ \mathscr{T} < C(k_*) + 3 \big\} \setminus \big\{ \mathscr{S}_{k_*} < \beta_1 \big\}, \quad \forall r \in [0,1]\,.$$

First, we observe that

$$(54) \qquad \mathscr{S}_{k_*}^{\gamma_m^r}(\xi) \;>\; c_{\mathscr{G}}(k_*) \;-\; \delta_m \;\geq\; \sup_{B^d} \mathscr{S}_{k_*}^{\gamma_m^r} \;-\; (C(k_*) + 2)\,\delta_m\,.$$

If m is large enough, then $(C(k_*) + 2)\,\delta_m < (\beta_{\partial \mathscr{V}} - \beta_1)$ and Lemma 3.6 implies that $\gamma_m^r(\xi) \notin \{ \mathscr{S}_{k_*} < \beta_1 \}$. As a by-product we get that $u_m^\xi|_{[0,r]}$ is a genuine flow line of $\Phi^{\mathscr{X}_{k_*}}$. Then, we estimate $\mathscr{T}(\gamma_m^r(\xi))$. We start by taking $r = 0$. In this case from (43) we get

$$\mathscr{T}(\gamma_m(\xi)) \;=\; \frac{\mathscr{S}_{k_m}^{\gamma_m}(\xi) - \mathscr{S}_{k_*}^{\gamma_m}(\xi)}{\delta_m} \;<\; \frac{c_{\mathscr{G}}(k_m) + \delta_m - c_{\mathscr{G}}(k_*) + \delta_m}{\delta_m} \;<\; C(k_*) + 2\,.$$

To prove the inequality for arbitrary r we bound the variation of $\mathscr{T}$ along $u_m^\xi|_{[0,r]}$ in terms of the action variation:

$$-\alpha_{k_*}(u_m^\xi|_{[0,r]}) \;=\; -\int_0^r \alpha_{k_*}\left(\frac{du_m^\xi}{d\rho}\right) d\rho \geq \int_0^r \left|\frac{du_m^\xi}{d\rho}\right|^2 d\rho$$

$$\geq \frac{1}{r}\left(\int_0^r \left|\frac{du_m^\xi}{d\rho}\right| d\rho\right)^2$$

$$\geq \frac{1}{r}\left(\int_0^r \frac{1}{1 + \sup_{\mathscr{H}} |d\mathscr{T}|} \left|\frac{d(\mathscr{T} \circ u_m^\xi)}{d\rho}\right| d\rho\right)^2$$

$$\geq \frac{1}{r(1 + \sup_{\mathscr{H}} |d\mathscr{T}|)^2} |\mathscr{T}(u_m^\xi(r)) - \mathscr{T}(u_m^\xi(0))|^2\,.$$

Using (52) and rearranging the terms we get for m large enough

$$|\mathscr{T}(\gamma_m^r(\xi)) - \mathscr{T}(\gamma_m(\xi))|^2 \;\leq\; r \cdot (1 + \sup_{\mathscr{H}} |d\mathscr{T}|)^2 \cdot (C(k_*) + 2)\,\delta_m \;<\; 1\,.$$

Hence, if m is large enough the bound on $\mathscr{T}$ we were looking for follows from

$$(55) \quad \mathscr{T}(\gamma_m^r(\xi)) \;\leq\; \mathscr{T}(\gamma_m(\xi)) \;+\; |\mathscr{T}(\gamma_m^r(\xi)) - \mathscr{T}(\gamma_m(\xi))| \;<\; (C(k_*) + 2) \;+\; 1\,.$$

The claim is thus completely established.

The last step to finish the proof of Theorem 3.1 is to show that $J_m^1 = \emptyset$ for m large enough. By contradiction, let $\xi \in J_m^1$. Since $\xi \in J_m^r$ for all $r \in [0,1]$, we see that u_m^ξ is a flow line of $\Phi^{\mathscr{X}_{k_*}}$ contained in $\{\mathscr{T} < C(k_*) + 3\} \setminus \{\mathscr{S}_{k_*} < \beta_1\}$. Using (52) and continuing the chain of inequalities in (48), we find

$$- \left(C(k_*) + 2\right) \delta_m \; < \; \alpha_{k_*}(u_m^\xi) \; \leq \; -\frac{\varepsilon_0^2}{\sqrt{1 + \varepsilon_0^2}}$$

(where we used that the real function $w \mapsto \frac{w}{\sqrt{1+w}}$ is increasing). Such inequality cannot be satisfied for m large, proving that the sets J_m^1 become eventually empty.

Finally, since $J_m^1 = \emptyset$, we obtain that $c_{\mathscr{G}}(k_*) \leq \sup_{B^d} \mathscr{S}_{k_*}^{\gamma_m^1} \leq c_{\mathscr{G}}(k_*) - \delta_m$. This contradiction finishes the proof of Theorem 3.1.

In the next section we will determine when η_k satisfies the hypotheses of the abstract theorem we have just proved.

4. Proof of the Main Theorem

We now move to the proof of points *(1)*, *(2)*, *(3)* of Theorem 1.6. In the first preparatory subsection, we will see when the action form is exact.

4.1. **Primitives for η_k.** We know that η_k is exact if and only if so is τ^σ. The next proposition, whose simple proof we omit, gives necessary and sufficient conditions for the transgression form to be exact.

Proposition 4.1. *If $[\widetilde{\sigma}] \neq 0$, then $\tau^\sigma|_{W_\nu^{1,2}}$ is not exact for any ν.*
If $[\widetilde{\sigma}] = 0$, then

$$\widehat{P}^\sigma : W_0^{1,2} \longrightarrow \mathbb{R}\,.$$

$$x \longmapsto \int_{B^2} \hat{u}_x^* \sigma$$

is a primitive for τ^σ. Here $\hat{u}_x$ is any capping disc for x. This definition extends the primitive P^σ, which we constructed on the subset of short loops.
If $[\widetilde{\sigma}]_b = 0$, then, given ν and a reference loop $x_\nu \in W_\nu^{1,2}$,

$$\widehat{P}^\sigma : W_\nu^{1,2} \longrightarrow \mathbb{R}\,.$$

$$x \longmapsto \int_{B^2} \hat{u}_{x_\nu, x}^* \sigma$$

is a primitive for τ^σ. Here $\hat{u}_{x_\nu, x}$ is a connecting cylinder from x_ν to x. If we take x_0 as a constant loop, the two definitions of $\widehat{P}^\sigma$ coincide on $W_0^{1,2}$.

Exercise 6. *Show that if $M = \mathbb{T}^2$ and $[\sigma] \neq 0$, then $\tau^\sigma|_{W_\nu^{1,2}}$ is not exact if $\nu \neq 0$.*

We set $\widehat{S}_k := A_k - \widehat{P}^\sigma \circ \mathrm{pr}_{W^{1,2}}$ in the two cases above where $\widehat{P}^\sigma$ is defined. Theorem A in [CIPP98] tells us when $\widehat{S}_k$ is bounded from below.

Proposition 4.2. *If $[\widetilde{\sigma}] = 0$, then $\widehat{S}_k : \Lambda_0 \to \mathbb{R}$ is bounded from below if and only if $k \geq c(L, \sigma)$. If $[\widetilde{\sigma}]_b = 0$, the same is true for $\widehat{S}_k : \Lambda_\nu \to \mathbb{R}$.*

Remark 4.3. *Originally the critical value was introduced by Mañé as the infimum of the values of k such that $\widehat{S}_k : \Lambda_0 \to \mathbb{R}$ is bounded from below [Mañ97, CDI97]. Thus, the proposition above establishes the equivalence between the more geometric definition in (4) and the original one.*

Exercise 7. *Prove that $\widehat{S}_k|_{\Lambda_\nu}$ is bounded from below if and only if $\widehat{S}_k|_{\Lambda_0}$ is bounded from below if and only if $\widehat{S}_k|_{\Lambda_0}$ is non-negative.*

As a by-product of Proposition 4.2, we can give a criterion guaranteeing that a vanishing sequence for η_k has bounded periods, provided $k > c(L, \sigma)$.

Corollary 4.4. *Let $\nu \in [\mathbb{T}, M]$ and $[\widetilde{\sigma}]_b = 0$. If $k > c(L, \sigma)$ and $b \in \mathbb{R}$, then there exists a constant $C(\nu, k, b)$ such that*

$$\forall\, \gamma \in \Lambda_\nu\,,\quad \widehat{S}_k(\gamma)\ <\ b\ \implies\ T\ <\ C(\nu, k, b)\,.$$

Proof. We readily compute

$$T\ =\ \frac{\widehat{S}_k(\gamma) - \widehat{S}_{c(L,\sigma)}(\gamma)}{k - c(L,\sigma)}\ \leq\ \frac{b - \inf_{\Lambda_\nu} \widehat{S}_{c(L,\sigma)}}{k - c(L,\sigma)}\ =:\ C(\nu, k, b)\,. \qquad \square$$

4.2. Non-contractible orbits. We now prove the existence of non-contractible orbits as prescribed by the Main Theorem.

Proof of Theorem 1.6.(1a). Let $\nu \in [\mathbb{T}, M]$ be a non-trivial class, σ be a magnetic form such that $[\sigma]_b = 0$ and $k > c(L, \sigma)$. Thanks to Proposition 4.2, the infimum of $\widehat{S}_k$ on Λ_ν is finite. Then, we apply Theorem 3.2 with $\mathscr{H} = \Lambda_\nu$ and $\widehat{\mathscr{S}} = \widehat{S}_k$ and we obtain a vanishing sequence $(\gamma_m)_{m \in \mathbb{N}}$ such that $\widehat{S}_k(\gamma_m)$ is uniformly bounded. By Corollary 4.4 the sequence of periods is bounded from above. By Corollary 2.12 the sequence of periods is also bounded away from zero. Therefore, we can apply Proposition 2.8 to get a limit point of the sequence. $\qquad \square$

4.3. Contractible orbits. We start by recalling a topological lemma.

Proposition 4.5. *If $d \geq 1$ and $\delta \leq \delta_*$ (see Lemma 2.10), there are natural bijections*

$$
\begin{array}{ccc}
\dfrac{\pi_{d+1}(M)}{\pi_1(M)} & \xrightarrow{\hspace{2cm}} & [S^{d+1},\, M] \\[2mm]
 & \searrow{\scriptstyle F} & \downarrow \\[2mm]
 & \left[\, (B^d, \partial B^d)\,,\ (W_0^{1,2}, M_0)\, \right] & \xrightarrow{\ i^{M_0}_{\{\ell < \delta\}}\ } \left[\, (B^d, \partial B^d)\,,\ (W_0^{1,2}, \{\ell < \delta\})\, \right]
\end{array}
$$

where $\pi_{d+1}(M)/\pi_1(M)$ is the quotient of $\pi_{d+1}(M)$ by the action of $\pi_1(M)$[1]. The trivial classes on the second line are identified with the class of constant maps in $[S^{d+1}, M]$ and with the class of the zero element in $\pi_{d+1}(M)/\pi_1(M)$.

Proof. The first horizontal map is $\frac{[\hat{u}]}{\pi_1(M)} \mapsto [\hat{u}]$. We leave as an exercise to the reader to show that is a bijection. The vertical map sends $[\hat{u}]$ to $[u]$, where u is defined as follows. Consider the equivalence relation $\sim$ on $B^d \times \mathbb{T}$:

$$(56)\quad (z_1, s_1) \sim (z_2, s_2) \qquad \Longleftrightarrow \qquad (z_1, s_1) = (z_2, s_2) \quad \vee \quad z_1 = z_2 \in \partial B^d\,.$$

---

[1]Here a choice of an arbitrary base point $q_0 \in M$ is to be understood: $\pi_{d+1}(M) := \pi_{d+1}(M, q_0)$ and $\pi_1(M) := \pi_1(M, q_0)$

If we interpret B^d as the unit ball in $\mathbb{R}^d$ and S^{d+1} as the unit sphere in $\mathbb{R}^{d+2}$ we can define the homeomorphism

$$Q : \frac{B^d \times \mathbb{T}}{\sim} \longrightarrow S^{d+1}$$

$$[z, s] \longmapsto \left(z, \sqrt{1 - |z|^2} \cdot e^{2\pi i s}\right),$$

where $e^{2\pi i s}$ belongs to $S^1 \subset \mathbb{R}^2$. We set $u(z)(s) := (\hat{u} \circ Q)([z, s])$. For a proof that the vertical map is well-defined and it is a bijection, we refer the reader to [Kli78, Proposition 2.1.7]. Finally, the second horizontal map is a bijection thanks to Lemma 2.10. $\qquad\square$

We can now prove the parts of the Main Theorem dealing with contractible orbits.

Proof of Theorem 1.6.(1b). Let $[\widetilde{\sigma}]_b = 0$, $k > c(L, \sigma)$ and fix some non-zero $\mathfrak{u} \in \pi_{d+1}(M)$, which exists by hypothesis. We apply Proposition 2.13 to the trivial interval $\{k\}$ and get the positive real numbers $\delta_{\{k\}}$, $b_{\{k\}}$ and $T_{\{k\}}$. Let
(57)
$$\Gamma_\mathfrak{u} := \left\{ \gamma = (x, T) : \left(B^d, \partial B^d\right) \longrightarrow \left(\Lambda_0, M_0 \times \{T_{\{k\}}\}\right) \ \middle| \ [x] \in F\big(\mathfrak{u}/\pi_1(M)\big) \right\}$$

By Proposition 4.5 we see that $\Gamma_\mathfrak{u} \in \left[(B^d, \partial B^d), (\Lambda_0, M_0 \times \{T_{\{k\}}\})\right]$ and that $i_{\mathscr{V}^{\delta_{\{k\}}}}^{M_0 \times \{T_{\{k\}}\}}(\Gamma_\mathfrak{u})$ is non-trivial. Therefore, we apply Theorem 3.1 with

$$\left[\begin{array}{lll} \mathscr{H} = \Lambda_0 & \mathscr{I} = \{k\} & \widehat{\mathscr{S}_k} = \widehat{S}_k \\ \beta_0 = b_{\{k\}}/2 & \mathscr{V} = \mathcal{V}^{\delta_{\{k\}}} & \mathscr{M} = M_0 \times \{T_{\{k\}}\} \\ \mathscr{G} = \Gamma_\mathfrak{u} & & \end{array} \right]$$

and we obtain a vanishing sequence $(\gamma_m)_{m \in \mathbb{N}}$ such that

$$\lim_{m \to +\infty} \widehat{S}_k(\gamma_m) = c_\mathfrak{u}(k) := \inf_{\gamma \in \Gamma} \sup_{B^d} \widehat{S}_k \circ \gamma \geq b_{\{k\}}.$$

The sequence of periods (T_m) is bounded from above by Corollary 4.4. The sequence (T_m) is also bounded away from zero by Corollary 2.12, since $\gamma_m \notin \{S_k < b_{\{k\}}/2\}$ for m big enough. Applying Proposition 2.8 we obtain a limit point of (γ_m). $\qquad\square$

Proof of Theorem 1.6.(2). Let $[\widetilde{\sigma}] = 0$ and fix $I = [k_0, k_1] \subset (e_0(L), c(L, \sigma))$. Let δ_I, b_I and T_I be as in Proposition 2.13. Fix $\gamma_0 \in M_0 \times \{T_I\}$ and $\gamma_1 \in \Lambda_0$ such that $\widehat{S}_{k_1}(\gamma_1) < 0$. Such element exists thanks to Proposition 4.2. Let $u_* : [0, 1] \to \Lambda$ be some path such that $u_*(0) = \gamma_0$ and $u_*(1) = \gamma(1)$ and denote by $[u_*] \in [(B^1, \partial B^1), (\Lambda_0, \{\gamma_0, \gamma_1\})]$ its homotopy class. By Proposition 2.13, γ_0 and γ_1 belong to different components of $\{\widehat{S}_{k_0} < b_I\}$. Thus, $i_{\{\widehat{S}_{k_0} < b_I\}}^{\{\gamma_0, \gamma_1\}}([u_*])$ is non-trivial. Therefore, we apply Theorem 3.1 with

$$\left[\begin{array}{lll} \mathscr{H} = \Lambda_0 & \mathscr{I} = I & \widehat{\mathscr{S}_k} = \widehat{S}_k \\ \beta_0 = b_I/2 & \mathscr{V} = \{\widehat{S}_{k_0} < b_I\} & \mathscr{M} = \{\gamma_0, \gamma_1\} \\ \mathscr{G} = [u_*] & & \end{array} \right]$$

and we get a vanishing sequence $(\gamma_m^k)_{m\in\mathbb{N}}$ with bounded periods, for almost every $k \in I$. Moreover, we have

$$\lim_{m\to+\infty} \widehat{S}_k(\gamma_m) \;=\; c_{[u_*]}(k) \;:=\; \inf_{u\in[u_*]} \sup_{B^1} \widehat{S}_k \circ u \;\geq\; b_I\,.$$

In particular, $\gamma_m^k \notin \{\widehat{S}_k < b_I/2\}$ for m large enough. Hence, the periods are bounded away from zero by Corollary 2.12. Now we apply Proposition 2.8 to get a limit point of (γ_m^k). Taking an exhaustion of $(e_0(L), c(L,\sigma))$ by compact intervals, we get a critical point for almost every energy in $(e_0(L), c(L,\sigma))$. □

Proof of Theorem 1.6.(3). Let $[\widetilde{\sigma}] \neq 0$ and fix $I = [k_0, k_1] \subset (e_0(L), +\infty)$. Let δ_I, b_I and T_I be as in Proposition 2.13. Since $[\widetilde{\sigma}] \neq 0$, there exists a non-zero $\mathfrak{u} \in \pi_2(M)$. We set

$$(58)\quad \Gamma_{\mathfrak{u}} := \Big\{ \gamma = (x, T) : (B^1, \partial B^1) \longrightarrow (\Lambda_0, M_0 \times \{T_I\}) \;\Big|\; [x] \in F(\mathfrak{u}/\pi_1(M)) \Big\}$$

By Proposition 4.5 we see that $\Gamma_{\mathfrak{u}} \in \big[(B^1, \partial B^1), (\Lambda_0, M_0 \times \{T_I\})\big]$ and that

$$i_{\mathcal{V}^{\delta_I}}^{M_0 \times \{T_I\}}(\Gamma_{\mathfrak{u}})$$

is non-trivial. Therefore, we apply Theorem 3.1 with

$$\left[\begin{array}{ccc} \mathscr{H} = \Lambda_0 & \mathscr{I} = I & \alpha_k = \eta_k \\[4pt] \beta_0 = b_I/2 & \mathscr{V} = \mathcal{V}^{\delta_I} & \mathscr{M} = M_0 \times \{T_I\} \\[4pt] \mathscr{G} = \Gamma_{\mathfrak{u}} & & \end{array}\right]$$

and we obtain a vanishing sequence $(\gamma_m^k)_{m\in\mathbb{N}} \subset \Lambda_0 \setminus \{S_k < b_I/2\}$ with bounded periods, for almost every $k \in I$. Since, the periods are bounded away from zero by Corollary 2.12, Proposition 2.8 yields a limit point of (γ_m^k), for almost every $k \in I$.

Taking an exhaustion of $(e_0(L), +\infty)$ by compact intervals, we get a contractible zero of η_k for almost every $k > e_0(L)$. □

5. Magnetic flows on surfaces I: Taĭmanov minimizers

In this and in the next section we are going to focus on the 2-dimensional case. Therefore, let us assume that M is a closed connected oriented surface. In this case $H^2(M;\mathbb{R}) \simeq \mathbb{R}$, where the isomorphism is given by integration and we identify $[\sigma]$ with a real number. Up to changing the orientation on M, we assume that $[\sigma] \geq 0$.

For simplicity, we are going to work in the setting of Section 1.6 and consider only purely kinetic Lagrangians. Namely, we take $L(q,v) = \frac{1}{2}|v|^2$, where $|\cdot|$ is induced by a metric g.

Since L depends only on g, we will use the notation (g,σ) where we previously used (L,σ). We readily see that $e_m(L) = e_0(L) = 0$ and that $c(g,\sigma) = 0$ if and only if $\sigma = 0$ (see Proposition 1.4). We recall that the periodic orbits with positive energy are parametrized by a positive multiple of the arc-length. Thus, they are immersed curve in M.

5.1. **The space of embedded curves.** The space of curves on a 2-dimensional manifold M has a particularly rich geometric structure. Observe, indeed, that for $n \geq 3$ the curves on M are generically embedded. On the other hand, if M is a surface, intersections between curves and self-intersections are generically stable. Therefore, one can refine the existence problem by looking at periodic orbits having a particular shape (see the beginning of Section 1.1 in [HS13] and references therein for a precise notion of the shape of a curve on a surface). For example, we consider the following question.

For which k and ν there exists a *simple* periodic orbit $\gamma \in \Lambda_\nu$ with energy $k > 0$?

Let us start by investigating the case $\nu = 0$. If $\gamma = (x, T)$ is a contractible simple curve, there exists an embedded disc $\hat{u} : B^2 \to M$ such that $\hat{u}(e^{2\pi i s}) = x(s)$. This map yields a path (u, T) in Λ_0 from a constant path (x_0, T), representing the centre of the disc, to (x, T). Integrating η_k along this path and summing the value of S_k at (x_0, T), we get

$$(59) \qquad \int_0^1 (u, T)^* \eta_k \ + \ S_k(x_0, T) \ = \ \frac{e(x)}{2T} \ + \ kT \ - \ \int_{B^2} \hat{u}^* \sigma \,.$$

Since $\hat{u}$ is an embedding, $\mathrm{area}(\hat{u}) \leq \mathrm{area}(M)$ and we find a uniform bound from below

$$(60) \quad \int_0^1 (u, T)^* \eta_k + S_k(x_0, T) \ \geq \ 0 + 0 - \sup_M |\sigma| \cdot \mathrm{area}(\hat{u}) \ \geq \ - \sup_M |\sigma| \cdot \mathrm{area}(M) \,.$$

This observation gives us the idea of defining a functional on the space of simple contractible loops and look for its global minima. First, we notice that $\int_{B^2} \hat{u}^* \sigma$ is invariant under an orientation-preserving change of parametrization. In order to make the whole right-hand side of (59) independent of the parametrization, we ask that $(\gamma, \dot{\gamma}) \in \Sigma_k$. This implies that

$$\sqrt{2k} \cdot T \ = \ \ell(x) \,, \qquad e(x) \ = \ \ell(x)^2 \,.$$

Substituting in (59), we get

$$(61) \qquad \int_0^1 (u, T)^* \eta_k \ + \ S_k(x_0, T) \ = \ \sqrt{2k} \cdot \ell(\partial D) \ - \ \int_D \sigma \ =: \ \mathcal{T}_k(D) \,,$$

where

$$D = [\hat{u}] \in \mathcal{D}(M) := \left\{ \begin{array}{c} \text{embeddings } \hat{u} : B^2 \longrightarrow M \,, \\ \text{up to orientation-preserving reparametrizations} \end{array} \right\}$$

and ∂D represents the boundary of D oriented in the counter-clockwise sense. We readily see that the critical points of this functional correspond to the periodic orbits we are looking for.

Proposition 5.1. *If D is a critical point of $\mathcal{T}_k : \mathcal{D}(M) \to \mathbb{R}$, then ∂D is the support of a simple contractible periodic orbit with energy k.*

In view of this proposition and the fact that $\mathcal{T}_k$ is bounded from below, we consider a minimizing sequence $(D_m)_{m \in \mathbb{N}} \subset \mathcal{D}(M)$. However, the sequence D_m might converge to a disc D_∞ which is not embedded. For example, D_∞ might have a self-tangency at some point q on its boundary (see Figure 1). However, in this case the support of D_∞ in M can be interpreted as an annulus A_∞ whose two boundary components touch exactly at q. Now we can resolve the singularity in the
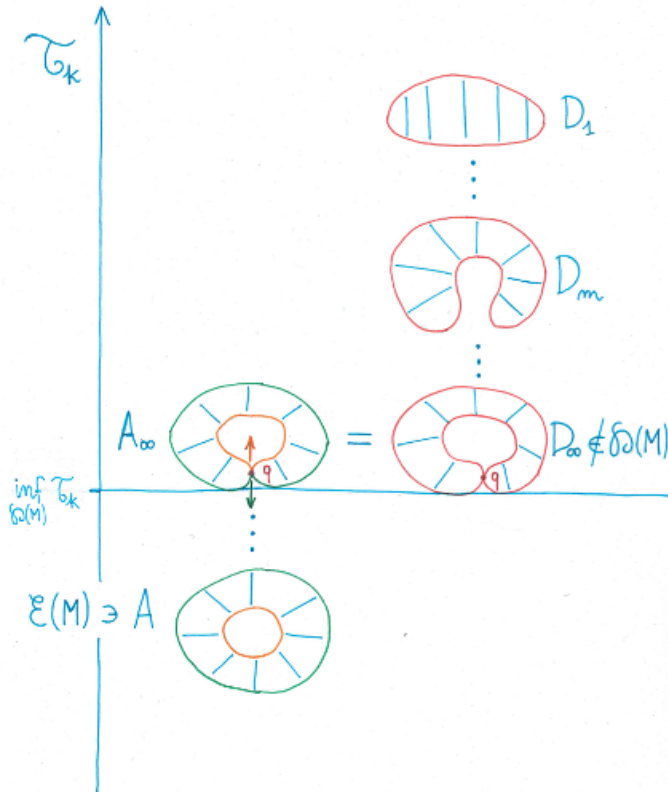
FIGURE 1. Minimizing sequence for $\mathcal{T}_k$ on $\mathcal{D}(M)$

space of annuli and get an embedded annulus A close to A_∞. The key observation is that $\mathcal{T}_k$ can be extended to the space of annuli and that

(62) $$\mathcal{T}_k(D_\infty) \; = \; \mathcal{T}_k(A_\infty) \; > \; \mathcal{T}_k(A) \,.$$

To justify the inequality in the passage above, we observe that $\ell(\partial A) < \ell(\partial A_\infty)$ from classic estimates in Riemannian geometry and that the contribution given by the integral of σ is of higher order. This heuristic argument prompts us to give the following definitions.

Definition 5.2. *Let $\mathcal{E}(M) = \{\,oriented\ embedded\ surfaces\ \Pi \to M\} \cup \{\emptyset\}$ and denote by $\mathcal{E}_+(M)$ and $\mathcal{E}_-(M)$ the surfaces having the same orientation as M and the opposite orientation, respectively. If $\Pi \in \mathcal{E}(M)$, then $\partial \Pi$ denotes the (possibly empty) multi-curve made by the boundary components of Π. If we define the length $\ell(\partial \Pi)$ as the sum of the lengths of the boundary components, we have a natural extension*

$$\mathcal{T}_k : \mathcal{E}(M) \; \longrightarrow \; \mathbb{R}$$
$$\Pi \; \longmapsto \; \sqrt{2k} \cdot \ell(\partial \Pi) \; - \; \int_\Pi \sigma \,.$$

As in (60) we find that $\mathcal{T}_k$ is bounded from below by $-\sup|\sigma| \cdot \mathrm{area}(M)$. Moreover, we observe that there is a bijection

(63)
$$\begin{array}{ccc} \mathcal{E}_+(M) & \longrightarrow & \mathcal{E}_-(M) \\ \Pi & \longmapsto & M \setminus \mathring{\Pi} \end{array} \qquad \text{such that} \qquad \mathcal{T}_k(M \setminus \mathring{\Pi}) \; = \; \mathcal{T}_k(\Pi) \; + \; \int_M \sigma \,.$$

Therefore, it is enough to look for a minimizer on $\mathcal{E}_-(M)$. The chain of inequalities (62) hints at the following result.

Proposition 5.3. *For all $k > 0$, there exists a minimizer Π^k of $\mathcal{T}_k|_{\mathcal{E}_-(M)}$. If $\partial \Pi^k = \{\gamma_i^k\}_i$, then the γ_i^k are periodic orbits with energy k.*

For a rigorous proof of this proposition we refer to [Taĭ93] and [CMP04]:

- In the former reference, Taĭmanov uses a finite dimensional reduction and works on the space of surfaces $\Pi \in \mathcal{E}(M)$ whose boundary is made by piecewise solutions of the twisted Euler-Lagrange equations with energy k.
- In the latter reference, the authors use a weak formulation of the problem on the space of integral currents $I_2(M) \supset \mathcal{E}(M)$.

In order to use Proposition 5.3 to prove the existence of periodic orbits with energy k, we have to ensure that $\partial \Pi^k \neq \emptyset$. To this purpose, we observe that $\partial \Pi^k = \emptyset$ implies $\Pi^k \in \{\emptyset, \overline{M}\}$, where $\overline{M}$ is M with the opposite orientation. We easily compute $\mathcal{T}_k(\emptyset) = 0$ and $\mathcal{T}_k(\overline{M}) = \int_M \sigma \geq 0$. Therefore, for every $k > 0$ we have

$$\inf_{\mathcal{E}_-(M)} \mathcal{T}_k \; \leq \; 0 \qquad \text{and} \qquad \left(\inf_{\mathcal{E}_-(M)} \mathcal{T}_k \; < \; 0 \quad \Longrightarrow \quad \partial \Pi^k \neq \emptyset \right).$$

Since the family of functionals $\mathcal{T}_k$ is monotone in k, we are led to define

(64)
$$\tau(g, \sigma) := \inf \left\{ k \mid \inf_{\mathcal{E}_-(M)} \mathcal{T}_k = 0 \right\}.$$

Proposition 5.4. *The value $\tau(g, \sigma)$ is a non-negative real number. Moreover,*

$$\tau(g, \sigma) \; > \; 0 \qquad \Longleftrightarrow \qquad \sigma_{q_0} \; < \; 0, \text{ for some } q_0 \in M \,.$$

If σ is exact, then

(65)
$$\tau(g, \sigma) \; = \; c_0(g, \sigma) := \inf_{d\theta=\sigma} \sup_{q \in M} |\theta_q| \,.$$

We leave the proof of the first statement of the proposition as an exercise to the reader. The second statement follows from [CMP04]. We can summarize our answer to the question raised at the beginning of this section with the following theorem.

Theorem 5.5. *Suppose that there exists $q_0 \in M$ such that $\sigma_{q_0} < 0$. Then, we can find a positive real number $\tau(g, \sigma)$, coinciding with $c_0(g, \sigma)$ when σ is exact, such that for every $k \in (0, \tau(g, \sigma))$, there exists a non-empty set of simple periodic orbits $\{\gamma_i^k\}$ having energy k and satisfying*

$$\sum_i [\gamma_i^k] \; = \; 0 \in H^1(M; \mathbb{Z}) \,.$$

6. MAGNETIC FLOWS ON SURFACES II: STABLE ENERGY LEVELS

In this last section we continue the study of twisted Lagrangian flows of kinetic type on surfaces by investigating the stability properties of their energy levels. To have a better geometric intuition, we are going to pull-back the twisted symplectic form to the tangent bundle. Thus, let $\flat : TM \to T^*M$ be the duality isomorphism given by g. We define the twisted tangent bundle as the symplectic manifold $(TM, \omega_{g,\sigma})$, where $\omega_{g,\sigma} := d(\flat^*\lambda) - \pi^*\sigma$. We readily see that $X_{(g,\sigma)}$ is the Hamiltonian flow of E with respect to the symplectic form $\omega_{g,\sigma}$. In this language, our problem is to understand when the hypersurface Σ_k is stable in the twisted tangent bundle. We will summarize the current knowledge on the subject in the following four propositions.

The first one sheds light on the relation between stability and the contact property in the generic case.

Proposition 6.1. *Let $k > 0$. If $[\sigma] \neq 0$ and $M = \mathbb{T}^2$, Σ_k is not of contact type. Moreover, if $X_{(g,\sigma)}|_{\Sigma_k}$ does not admit any non-trivial integral of motion, then:*

(1) *If $[\sigma] = 0$ or $M \neq \mathbb{T}^2$ and $[\sigma] \neq 0$, Σ_k is stable if and only if it is of contact type.*

(2) *If $M = \mathbb{T}^2$ and $[\sigma] \neq 0$, every stabilizing form on Σ_k is closed and it has non-vanishing integral over the fibers of π.*

The second proposition gives obstruction to the contact property.

Proposition 6.2. *The following statements hold true.*

(1) *If $[\sigma] = 0$, then Σ_k is not of negative contact type.*

(2) *If $[\sigma] \neq 0$, then*

    (a) *if $M = S^2$, Σ_k is not of negative contact type;*

    (b) *if M has genus higher than 1, there exists $c_h(g, \sigma) > 0$ such that*

        • *Σ_k is not of negative contact type, when $k > c_h(g, \sigma)$;*

        • *$\Sigma_{c_h(g,\sigma)}$ is not of contact type;*

        • *Σ_k is not of positive contact type, when $k < c_h(g, \sigma)$;*

The third proposition deals with positive results on stability.

Proposition 6.3. *The following statements hold true.*

(1) *If $[\sigma] = 0$, Σ_k is of contact type if $k > c_0(g, \sigma)$. If $M = \mathbb{T}^2$, for every Riemannian metric g there exists an exact form σ_g for which $\Sigma_{c_0(g,\sigma_g)}$ is of contact type.*

(2) *If $[\sigma] \neq 0$ and $M \neq \mathbb{T}^2$, Σ_k is of contact type for k big enough.*

(3) *If σ is a symplectic form on M, then Σ_k is stable for k small enough.*

The last proposition deals with negative results on stability.

Proposition 6.4. *The following statements hold true.*

(1) *If $[\sigma] = 0$ and $M \neq \mathbb{T}^2$, Σ_k is not of contact type, for $k < c_0(g, \sigma)$;*

(2) *If $[\sigma] \neq 0$ and there exists $q \in M$ such that $\sigma_q < 0$, then*

    (a) *when $M \neq \mathbb{T}^2$, Σ_k is not of contact type, for k low enough;*

    (b) *when $M = \mathbb{T}^2$, Σ_k does not admit a closed stabilizing form, for k low enough.*

(3) *If $M = S^2$, there exists an energy level associated to some $\overline{g}$ and some everywhere positive form $\overline{\sigma}$, which is not of contact type.*

Before embarking in the proof of such propositions, we make the following observation.

Lemma 6.5. *Let $k > 0$ and set $s := 1/\sqrt{2k}$. Then, the flows of $\Phi^{(g,\sigma)}|_{\Sigma_k}$ and $\Phi^{(g,s\sigma)}|_{\Sigma_{1/2}}$ are conjugated up to a time reparametrization.*

Proof. By Section 1.6 we know that the projections to M of the trajectories of $\Phi^{(g,\sigma)}|_{\Sigma_k}$ and of $\Phi^{(g,s\sigma)}|_{\Sigma_{1/2}}$ both satisfy the equation $\kappa_\gamma = s \cdot f(\gamma)$. Therefore, if

$$t \longmapsto \left(\gamma(t), \frac{d\gamma}{dt}(t) \right)$$

is a trajectory of the former flow and we set $\gamma_s(t') := \gamma(st')$, then

$$t' \longmapsto \left(\gamma_s(t'), \frac{d\gamma_s}{dt'}(t') \right) = \left(\gamma(st'),\ s \cdot \frac{d\gamma}{dt}(st') \right)$$

is a trajectory of the latter flow. $\qquad\square$

Therefore, given (g, σ), instead of studying the flow $\Phi^{(g,\sigma)}$ on each energy level Σ_k, we can study the 1-parameter family of flows $\Phi^{(g,s\sigma)}$ on $SM := \Sigma_{1/2}$ as s varies in $(0, +\infty)$. The advantage of rescaling σ is that now we can work on a fixed three-dimensional manifold: SM. The tangent bundle of SM has a global frame (X, V, H) and corresponding dual co-frame (α, ψ, β), which we now define.

Let $\mathcal{H} \subset SM$ be the horizontal distribution given by the Levi-Civita connection of g. For every $(q, v) \in SM$, $X_{(q,v)}$ and $H_{(q,v)}$ are defined as the unique elements in $\mathcal{H}$ such that

$$d_{(q,v)}\pi\big(X_{(q,v)}\big) = v, \qquad d_{(q,v)}\pi\big(H_{(q,v)}\big) = \imath \cdot v.$$

Analogously, $\alpha_{(q,v)}$ and $\beta_{(q,v)}$ are defined by

$$\alpha_{(q,v)}(\cdot) = g_q\big(v, d_{(q,v)}\pi(\cdot)\big), \qquad \beta_{(q,v)}(\cdot) = g_q\big(\imath \cdot v, d_{(q,v)}\pi(\cdot)\big).$$

The vector V is the generator of the rotations along the fibers $\varphi \mapsto (q, \cos\varphi\, v + \sin\varphi\, \imath \cdot v)$. The form ψ is the connection 1-form of the Levi-Civita connection. If $W \in T_{(q,v)}SM$ and $w(t) = (\gamma(t), v(t))$ is a curve such that $w(0) = (q, v)$ and $\dot{w}(0) = W$, then

$$\psi_{(q,v)}(W) = g_q\big(\nabla_{\dot\gamma(0)}v, \imath \cdot v\big).$$

Finally, we orient SM using the frame (X, V, H).

The proof of the following proposition giving the structural relations for the co-frame is a particular case of the identities proven in [GK02].

Proposition 6.6. *Let K be the Gaussian curvature of g. We have the relations:*

(66) $\qquad d\alpha = \psi \wedge \beta, \qquad d\psi = K\beta \wedge \alpha = -K\pi^*\mu, \qquad d\beta = \alpha \wedge \psi.$

Using the frame (X, V, H) we can write

$$X_s := X_{(g,s\sigma)} = X + sfV, \qquad \omega_s := \omega_{g,s\sigma}|_{SM} = d\alpha - s\pi^*\sigma.$$

We also use the notation Φ^s for the flow of X_s on SM.

6.1. **Stability of the homogeneous systems.** Let us start by describing the stability properties of the homogeneous examples introduced in Section 1.6.

6.1.1. *The two-sphere.* In this case we have $\sigma = \mu = K\mu$. Hence,

$$\omega_s = d\alpha - s\pi^*\sigma = d(\alpha + s\psi) \quad \text{and} \quad (\alpha + s\psi)(X_s) = (\alpha + s\psi)(X + sV) = 1 + s^2.$$

Every energy level is of positive contact type.

6.1.2. *The two-torus.* In this case we compute

$$d\psi = K\mu = 0 \quad \text{and} \quad \psi(X_s) = \psi(X + sV) = s.$$

Every energy level is stable.

6.1.3. *The hyperbolic surface.* In this case we have $\sigma = \mu = -K\mu$. Hence,

$$\omega_s = d\alpha - s\pi^*\sigma = d(\alpha - s\psi) \quad \text{and} \quad (\alpha - s\psi)(X_s) = (\alpha - s\psi)(X + sV) = 1 - s^2.$$

Every energy level Σ_k with $k > \frac{1}{2}$ is of positive contact type. Every energy level Σ_k with $k < \frac{1}{2}$ is of negative contact type. As follows from Proposition 6.2, $c_h(g, \sigma) = 1/2$ and $\Sigma_{1/2}$ is not stable.

6.2. Invariant measures on SM.

A fundamental ingredient in the proof of the four propositions is the notion of invariant measure for a flow. In this subsection, we recall this notion and we observe that twisted systems of purely kinetic type always possess a natural invariant measure called the *Liouville measure*.

Definition 6.7. *A Borel measure ξ on SM is Φ^s−invariant, if $\xi(\Phi_t^s(A)) = \xi(A)$, for every $t \in \mathbb{R}$ and every Borel set A. This is equivalent to asking*

$$(67) \qquad \int_{SM} dh(X_s)\, \xi = 0, \qquad \forall\, h \in C^\infty(SM, \mathbb{R}).$$

The rotation vector *of ξ is $\rho(\xi) \in H_1(SM, \mathbb{R})$ defined by duality on $[\tau] \in H^1(SM, \mathbb{R})$:*

$$(68) \qquad < [\tau], \rho(\xi) > = \int_{SM} \tau(X_s)\, \xi,$$

where $\tau \in \Omega^1(SM)$ is any closed form representing the class $[\tau]$.

Since X_s is a section of $\ker \omega_s$ and ω_s is nowhere vanishing, we can find a unique volume form Ω_s such that $\imath_{X_s}\Omega_s = \omega_s$. We can write $\Omega_s = \tau_s \wedge \omega_s$, where τ_s is any 1-form such that $\tau_s(X_s) = 1$. We easily see that $\alpha(X + sfV) = 1 + 0$. Hence, $\Omega_s = \alpha \wedge \omega_s = \alpha \wedge d\alpha$. Notice, indeed, that $\alpha \wedge \pi^*\sigma = 0$ since it is annihilated by V.

Definition 6.8. *The* Liouville measure *ξ_{SM} on SM is the Borel measure defined by integration with the differential form $\alpha \wedge d\alpha$. It is an invariant measure for Φ^s for every $s > 0$.*

In order to compute the rotation vector of ξ_{SM}, we need a lemma which tells us when ω_s is exact. The easy proof is left to the reader.

Lemma 6.9. *If σ is exact, then $\pi^*\sigma$ is exact and we have an injection*

$$(69) \qquad \begin{array}{c} \textit{Primitives of } \sigma \longrightarrow \textit{Primitives of } \omega_s \\ \zeta \longmapsto \alpha - s\pi^*\zeta. \end{array}$$

If $M \neq \mathbb{T}^2$, then $\pi^* \sigma$ is exact and we have an injection

(70)
$$Primitives \ of \ \sigma - \frac{[\sigma]}{2\pi\chi(M)} K\mu \ \longrightarrow \ Primitives \ of \ \omega_s$$

$$\zeta \ \longmapsto \ \alpha - s\pi^*\zeta + s\frac{[\sigma]}{2\pi\chi(M)}\psi \, .$$

If $M = \mathbb{T}^2$ and σ is non-exact, then ω_s is non-exact.

We can now state a proposition concerning $\rho(\xi_{SM})$.

Proposition 6.10. *If $[\sigma] \neq 0$ and $M = \mathbb{T}^2$, then there holds $\rho(\xi_{SM}) = s[\sigma] \cdot [S_q M]$, where $[S_q M] \in H_1(SM, \mathbb{Z})$ is the class of a fiber of $SM \to M$ oriented counter-clockwise. Otherwise, $\rho(\xi_{SM}) = 0$.*

Proof. Let $[\tau] \in H^1(SM; \mathbb{R})$. We notice that

$$\tau(X_s) \, \alpha \wedge d\alpha \ = \ \imath_{X_s}\Big(\tau \wedge \alpha \wedge d\alpha\Big) \ + \ \tau \wedge \imath_{X_s}\big(\alpha \wedge d\alpha\big) \ = \ 0 \ + \ \tau \wedge \omega_s \, .$$

Therefore,

$$< [\tau], \rho(\xi_{SM}) > = \ \int_{SM} \tau \wedge \omega_s \ = \ s \int_{SM} \tau \wedge \pi^*\sigma \, .$$

If $M = \mathbb{T}^2$, then $S\mathbb{T}^2 \simeq S^1 \times \mathbb{T}^2$ and we can use Fubini's theorem to integrate separately in the vertical directions and in the horizontal direction. Observe that since τ is closed, the integral over a fiber $S_q\mathbb{T}^2$ does not depend on q. Thus we find

$$\int_{S\mathbb{T}^2} \tau \wedge \pi^*\sigma \ = \ < [\tau], [S_q\mathbb{T}^2] > \cdot [\sigma] \, .$$

and the proposition is proven for the 2-torus. When $M \neq \mathbb{T}^2$, $\pi^*\sigma$ is exact and, therefore, $\int_{SM} \tau \wedge \pi^*\sigma = 0$. The proposition is proven also in this case. $\square$

We now proceed to the proofs of the four propositions.

6.3. **Proof of Proposition 6.1.** If $M = \mathbb{T}^2$ and $[\sigma] \neq 0$, then ω_s is not exact by Lemma 6.9. In particular, SM cannot be of contact type. This proves the first statement of the proposition. Now let $\tau_s \in \Omega^1(SM)$ be a stabilizing form for ω_s. Since $\ker(d\tau_s) \supset \ker \omega_s$, there exists a function $\rho_s : SM \to \mathbb{R}$ such that $d\tau_s = \rho_s \omega_s$. Taking the exterior differential in this equation, we get $0 = d\rho_s \wedge \omega_s$. Plugging in the vector field X_s we get $0 = d\rho_s(X_s)\omega_s$. Since ω_s is nowhere zero, we conclude that $d\rho_s(X_s) = 0$. Namely, ρ_s is a first integral for the flow. By assumption, ρ_s is equal to a constant. If $\rho_s = 0$, then τ_s is closed, if $\rho_s \neq 0$, then τ_s is a contact form. Suppose the first alternative holds. Since $\tau_s(X_s) \neq 0$ everywhere, we have

$$0 \ \neq \ \int_{SM} \tau_s(X_s)\xi_{SM} \ = \ < [\tau_s], \rho(\xi_{SM}) > \, .$$

By Proposition 6.10, this can only happen if $M = \mathbb{T}^2$ and $< [\tau_s], [S_q\mathbb{T}^2] > \neq 0$, which is what we had to prove.

6.4. **Proof of Proposition 6.2.** The proof of the second proposition is based on the fact that when ω_s is exact we can associate a number to every invariant measure with zero rotation vector.

Definition 6.11. *Suppose ω_s is exact and that ξ is a Φ^s-invariant measure with $\rho(\xi) = 0$. We define the* action *of ξ as the number*

$$(71) \qquad \mathcal{S}_s(\xi) := \int_{SM} \tau_s(X_s)\,\xi\,,$$

where τ_s is any primitive for ω_s. Such number does not depend on τ_s since $\rho(\xi) = 0$.

The action of invariant measures gives an obstruction to being of contact type.

Lemma 6.12. *Suppose ω_s is exact and that ξ is a non-zero Φ^s-invariant measure with $\rho(\xi) = 0$. If $\mathcal{S}_s(\xi) \leq 0$, then SM cannot be of positive contact type. If $\mathcal{S}_s(\xi) \geq 0$, then SM cannot be of negative contact type.*

Proof. If SM is of positive contact type, there exists τ_s such that $d\tau_s = \omega_s$ and $\tau_s(X_s) > 0$. Therefore,

$$\mathcal{S}_s(\xi) = \int_{SM} \tau_s(X_s)\,\xi \geq \inf_{SM} \tau_s(X_s) \cdot \xi(SM) > 0\,.$$

For the case of negative contact type, we argue in the same way. $\qquad\square$

Let us now compute the action of the Liouville measure.

Proposition 6.13. *If σ is exact, then*

$$(72) \qquad \mathcal{S}_s(\xi_{SM}) = \xi_{SM}(SM) = 2\pi[\mu]\,.$$

If $M \neq \mathbb{T}^2$, then

$$(73) \qquad \mathcal{S}_s(\xi_{SM}) = \xi_{SM}(SM) + s^2 \frac{[\sigma]^2}{\chi(M)}\,.$$

Proof. If $\sigma = d\zeta$, then $\alpha - s\pi^*\zeta$ is a primitive of ω_s by Lemma 6.9 and we have

$$(74) \qquad (\alpha - s\pi^*\zeta)(X_s)_{(q,v)} = 1 - s\zeta_q(v)\,, \quad \forall\,(q,v) \in SM\,.$$

Consider the *flip* $I : SM \to SM$ given by $I(q,v) := (q,-v)$. We see that

$$(I^*\alpha)_{(q,v)} = \alpha_{I(q,v)}dI = g_q(-v, d\pi dI\cdot) = \alpha_{I(q,v)}\,.$$

Hence ξ_{SM} is I-invariant. However, $\zeta \circ I(q,v) = -\zeta(q,v)$. Therefore,

$$(75) \qquad \int_{SM} \zeta\,\xi_{SM} = 0$$

and from the definition of action given in (71), we see that (72) is satisfied. To prove the second identity, we consider a primitive $\alpha - s\pi^*\zeta + s\frac{[\sigma]}{2\pi\chi(M)}\psi$ for ω_s as prescribed by Lemma 6.9. We compute

$$(76) \qquad \left(\alpha - s\pi^*\zeta + s\frac{[\sigma]}{2\pi\chi(M)}\psi\right)(X_s)_{(q,v)} = 1 - s\zeta_q(v) + s^2\frac{[\sigma]}{2\pi\chi(M)}f(q)\,.$$

Thus, we need to estimate the integral of $f \circ \pi$ on SM. Let U_i be an open cover of M such that $SU_i \simeq S^1 \times U_i$ and let a_i be a partition of unity subordinated to it.

We have

$$\int_{SM} f(q)\,\alpha \wedge d\alpha \;=\; \int_{SM} f(q)\,\alpha \wedge \psi \wedge \beta \;=\; -\int_{SM} f(q)\,\psi \wedge \pi^*\mu$$
$$= -\sum_i \int_{SU_i} a_i(q)\,\psi \wedge \pi^*\sigma$$
$$= -\sum_i \int_{S^1 \times U_i} a_i(q)\,(-d\varphi \wedge \pi^*\sigma)$$
$$= \sum_i \int_{U_i} a_i(q)\,\sigma \int_{S^1} d\varphi$$
$$= 2\pi \sum_i \int_{U_i} a_i(q)\,\sigma$$
$$= 2\pi[\sigma]\,,$$

where φ is an angular coordinate on S_qU_i going in the clockwise direction (hence the presence of an additional minus sign in the third line). Putting this computation together with (75), we get the desired identity. $\qquad\square$

Proposition 6.2 now follows from Lemma 6.12 and Proposition 6.13 after defining

$$(77) \qquad c_h(g,\sigma) := -\frac{[\sigma]^2}{4\pi\chi(M)[\mu]}\,, \qquad \text{when } M \text{ has genus higher than one}\,.$$

Remark 6.14. *We have seen in the homogeneous example above that $c_h(g,\sigma) = c(g,\sigma)$. The relation between c_h and the Mañé critical value was studied in general by G. Paternain in [Pat09]. There the author proves that $c_h(g,\sigma) \leq c(g,\sigma)$ and that $c_h(g,\sigma) = c(g,\sigma)$ if and only if g is a metric of constant curvature and σ is a multiple of the area form.*

6.5. **Proof of Proposition 6.3.** Suppose that σ is exact and let us consider a primitive $\alpha - s\pi^*\zeta$ given by Lemma 6.9. We have

$$(\alpha \;-\; s\pi^*\zeta)(X_s)_{(q,v)} \;=\; 1 \;-\; s\zeta_q(v) \;\geq\; 1 \;-\; s\sup_M|\zeta|\,.$$

Requiring that the right hand-side is positive is equivalent to saying that

$$k \;=\; \frac{1}{2s^2} \;>\; \sup_M \frac{1}{2}|\zeta|^2\,.$$

Since this holds for every ζ which is a primitive for σ, we have that the last inequality is equivalent to $k > c_0(g,\sigma)$. Contreras, Macarini and G. Paternain also found in [CMP04] examples of exact systems on $\mathbb{T}^2$, which are of contact type for $k = c_0(g,\sigma)$ (see also [Ben14a, Section 4.1.1]). We will not discuss these examples here and we refer the reader to the cited literature for more details.

Let us now deal with the non-exact case. If $M \neq \mathbb{T}^2$, then we consider a primitive of the form $\alpha - s\pi^*\zeta + s\frac{[\sigma]}{2\pi\chi(M)}\psi$ and we compute

$$(78) \quad \left(\alpha \;-\; s\pi^*\zeta \;+\; s\frac{[\sigma]}{2\pi\chi(M)}\psi\right)(X_s)_{(q,v)} \;=\; 1 \;-\; s\zeta_q(v) \;+\; s^2\frac{[\sigma]}{2\pi\chi(M)}f(q)\,.$$

We can give the estimate from below

$$1 \; - \; s\zeta_q(v) \; + \; s^2 \frac{[\sigma]}{2\pi\chi(M)} f(q) \; \geq \; 1 \; - \; s\sup_M |\zeta| - s^2 \left| \frac{[\sigma]}{2\pi\chi(M)} \right| \cdot \sup_M |f|$$

and we see that this quantity is strictly positive for s small enough.

Suppose now that σ is a symplectic form on M. We have three cases.

(1) If $M = S^2$, then the quantity in (78) is bounded from below by

$$1 - s\sup_M |\zeta| + s^2 \frac{[\sigma]}{4\pi} \cdot \inf_M f \, .$$

Since $[\sigma] > 0$, we have that $\inf f > 0$ and we see that such quantity is strictly positive for big s.

(2) If M has genus larger than 1, then the quantity in (78) is bounded from above by

$$1 \; + \; s\sup_M |\zeta| \; + \; s^2 \frac{[\sigma]}{2\pi\chi(M)} \cdot \inf_M f \, .$$

Since $\chi(M) < 0$ and $\inf f > 0$, such quantity is strictly negative for big s.

(3) If $M = \mathbb{T}^2$, then there exists a closed form $\tau \in \Omega^1(S\mathbb{T}^2)$ such that $\tau(V) = 1$ (prove such statement as an exercise). Thus, we get

$$(79) \qquad \tau(X_s) \; = \; \tau(X) \; + \; sf \; \geq \; \inf_{SM} \tau(X) \; + \; s\inf_M f$$

and such quantity is positive provided $\inf f > 0$ and s is big enough.

6.6. **Proof of Proposition 6.4.** If σ is exact and $k < c_0(g,\sigma)$, we can use Theorem 5.5 to find an embedded surface $\Pi \subset M$ with non-empty boundary $\partial\Pi = \{\gamma_i\}$ such that $\mathcal{T}_k(\Pi) < 0$ and the γ_i's are periodic orbits of Φ^s (parametrized by arc-length). Let $(\gamma_i, \dot\gamma_i)$ be the corresponding curve on SM and let ξ_i be the associated invariant measure. Define $\xi_{\partial\Pi} := \sum_i \xi_i$. What is its rotation vector? Call $\pi_* : H_1(SM;\mathbb{R}) \to H_1(M;\mathbb{R})$ the map induced by the projection π in homology and observe that

$$(80) \qquad \pi_*(\rho(\xi_{\partial\Pi})) \; = \; \sum_i \pi_*(\rho(\xi_i)) \; = \; \sum_i [\gamma_i] \; = \; [\partial\Pi] \; = \; 0 \, .$$

Exercise 8. *The map π_* is an isomorphism if $M \neq \mathbb{T}^2$.*

Thus, we conclude that $\rho(\xi_{\partial\Pi}) = 0$, if $M \neq \mathbb{T}^2$. Let us compute the action in this case. As before, we use a primitive $\alpha - s\pi^*\zeta$:

$$\mathcal{S}_s(\xi_{\partial\Pi}) = \sum_i \int_{SM} (1 - s\zeta_q(v))\xi_i \; = \; \sum_i \int_0^{\ell(\gamma_i)} \big(1 - s\zeta_{\gamma_i}(\dot\gamma_i)\big)\mathrm{d}t$$

$$(81) \hspace{4cm} = \; \sum_i \ell(\gamma_i) - s\int_0^{\ell(\gamma_i)} \gamma_i^*\zeta$$

$$\hspace{4cm} = \; \ell(\partial\Pi) - s\int_\Pi \sigma = s\mathcal{T}_k(\Pi) \, .$$

By hypothesis the last quantity is negative and Lemma 6.12 tells us that Σ_k cannot be of positive contact type. Since by Proposition 6.2, Σ_k cannot be of negative contact type either, point *(1)* of the proposition is proved.

We now move to prove point *(2a)* with the aid of a little exercise.

Exercise 9. *We prove a generalization of* (81), *when* $M \neq \mathbb{T}^2$. *Let* Π *be an embedded surface such that* $\partial \Pi$ *is a union of periodic orbits and let* $\xi_{\partial \Pi}$ *be the invariant measure constructed as before. Then,*

$$(82) \qquad \frac{\mathcal{S}_s(\xi_{\partial \Pi})}{s} \;=\; \mathcal{T}_k(\Pi) \;+\; \frac{\mathfrak{o}(\Pi)\chi(\Pi)[\sigma]}{\chi(M)} \,,$$

where $\mathfrak{o}(\Pi) \in \{+1, -1\}$ *record the orientation of* Π. *To prove such identity one recalls that* $\kappa_{\gamma_i} = sf(\gamma_i)$ *and then uses the Gauss-Bonnet theorem (taking into account orientations) to express the integral of the geodesic curvature along* $\partial \Pi$. *What happens if we consider* $M \setminus \Pi$? *Do the two expressions for* $\mathcal{S}_s(\xi_{\partial \Pi})$ *agree? Remember relation* (63).

The problem with formula (82) is that Theorem 5.5 does not give any information on the Euler characteristic of Π. To circumvent this problem we need the following result by Ginzburg [Gin87] (see also [AB15, Chapter 7]).

Proposition 6.15. *If* $\sup f > \varepsilon$ *for some* $\varepsilon < 0$, *there exists a constant* $C > 0$ *such that for every small enough* k *we can find a simple periodic orbit* γ_+^k *supported on* $\{f > \varepsilon\}$ *and such that* $\ell(\gamma_+^k) \leq \sqrt{2k}C$.
If $\inf f < -\varepsilon$, *for some* $\varepsilon > 0$, *there exists* $C > 0$ *such that for every small enough* k, *there exists a simple periodic orbit* γ_-^k *supported on* $\{f < -\varepsilon\}$ *and such that* $\ell(\gamma_-^k) \leq \sqrt{2k}C$.

If f is negative at some point, by Proposition 6.15, there exists γ_-^k with the properties listed above, for k small. In particular, γ_-^k bounds a small disc D_-^k. Since the geodesic curvature of γ_-^k is very negative, such disc lies in $\mathcal{E}_-(M)$. When $M \neq \mathbb{T}^2$, we use (82) and find

$$\frac{S_s(\xi_{\partial D_-^k})}{s} \;=\; \mathcal{T}_k(D_-^k) \;-\; \frac{2}{\chi(M)}[\sigma] \,.$$

By the estimate on the length of γ_-^k we get that $|\mathcal{T}_k(D_-^k)| \leq Ck^2$ (see (21)). Therefore, $S_s(\xi_{\partial D_-^k})$ has the opposite sign of $\chi(M)$ for k small enough. Combining Lemma 6.12 and Proposition 6.2, point *(2a)* is proven.

Let us deal now with the case of the 2-torus. Since $[\sigma] > 0$, by Proposition 6.15 there exists also γ_+^k bounding a disc D_+^k. Let $\Pi^k = D_-^k \cup D_+^k$. We claim that the measure $\xi_{\partial \Pi^k}$ has zero rotation vector.

Exercise 10. *Prove the claim by showing that* $(\gamma_+^k, \dot\gamma_+^k)$ *is freely homotopic in* $S\mathbb{T}^2$ *to* $[S_q\mathbb{T}^2]$, *namely the class of a fiber with orientation given by* V. *Analogously, prove that* $(\gamma_-^k, \dot\gamma_-^k)$ *is freely homotopic to a fiber with the opposite orientation.*

If τ_s is a closed stabilizing form, we have that the function $\tau_s(X_s)$ is nowhere zero. Therefore,

$$0 \;\neq\; \int_{S\mathbb{T}^2} \tau_s(X_s)\, \xi_{\partial \Pi^k} \;=\; <[\tau_s], \rho(\xi_{\partial \Pi^k})> \;=\; 0 \,,$$

which is a contradiction.

We omit the proof of point *(3)*, for which we refer the reader to [Ben14b].

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