Publicaciones Matemáticas del Uruguay
Volumen 16, Julio 2016, Páginas 127–167
S 0797-1443

INTRODUCTION TO NON-UNIFORM AND PARTIAL HYPERBOLICITY

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Abstract. These are notes for a minicourse given at Regional Norte UdelaR in Salto, Uruguay for the conference "CIMPA Research School - Hamiltonian and Langrangian Dynamics". The purpose of the notes is to present the theory of non-uniformly hyperbolic diffeomorphisms trying to concentrate in some simplified contexts and explain some of the main techniques in the field. Some of the topics include: Lyapunov exponents, Invariant manifolds (Pesin theory and persistence properties) and dynamical consequences. The topics will help introduce some concepts for the second part of the minicourse given by M.C. Arnaud but will also cover some topics of independent interest.

1. Introduction

The dynamics of uniformly hyperbolic systems is by now quite well understood in many aspects; for example: the spectral decomposition theorem allows one to decompose the dynamics in basic pieces which admit a quite precise coding (via Markov partitions) and the thermodynamical formalism provides information on the ergodic properties of invariant measures which have relevant dynamical or geometric meaning (see [Sh, KH], for example).

Of course, the understanding of uniformly hyperbolic systems is not complete, but there are many reasons for considering weaker forms of hyperbolicity. An important reason is that conservative dynamics are rarely uniformly hyperbolic[1].

There are essentially two ways to weaken uniform hyperbolicity: one consists on weakening the uniformity, by allowing to see hyperbolicity in almost every orbit but so that to see the hyperbolicity one has to "wait" a different amount of time depending on the point (this is called *non-uniform hyperbolicity*); the other consists in retaining the uniformity, but weakening the hyperbolicity by allowing certain bundles to be neutral yet "dominated" by the uniformly hyperbolic ones (this is called *partial hyperbolicity*).

In this notes, we pretend to give a unified view of this two generalizations by trying to study the dynamics from a local point of view, building charts around each point and considering the dynamics of sequences of diffeomorphisms of an Euclidean space. The main results we present have to do with the construction of invariant manifolds and the point of view is to try first to explain the (easier) case of periodic points and then try to convince the reader that the arguments go through

[1]In the conservative setting, being uniformly hyperbolic is the same as being Anosov, and it is well know that this imposes several strong restrictions on the topology of the manifold and isotopy class of the diffeomorphism (see [KH]). Moreover, there are also some local obstructions (such as possessing totally elliptic periodic points).

in these more general settings albeit some heavier notation and some adjustments on the statements.

This text has a strong subjective selection of topics and it is by no means a survey of the subject. It is intended as a first introduction to these topics which should be then complemented and deepened by the use of the standard references such as [KH, Sh, HPS] or others. Even if the text lacks a complete presentation of results, we have tried to provide at least a glimpse on further developments and ramifications of the subject. This choice has been even more subjective and depends heavily on the taste of the author.

1.1. **Organization of the notes.** In section 2 we give some preliminaries on ergodic theory which are relevant to what follows; in particular we provide a sketch of the proof of Oseledet's theorem in dimension 2. In this section we start to show the analogies between periodic orbits and ergodic measures.

In section 3 we show how one can pass the information on the tangent dynamics back to the manifold. This is probably the most important section of the notes and where the proof of the stable manifold theorem for periodic points is done in quite some detail and then the study of Pesin's charts and manifolds is explained. In section 4 we give a glimpse on the classical theory of non-uniform hyperbolicity and in section 5 we do the same with partial hyperbolicity and dominated splittings.

Finally, we end in section 6 presenting some applications of the previous result and explaining a recent result joint with Sylvain Crovisier and Martín Sambarino dealing with the geometry of partially hyperbolic attractors.

1.2. **Acknowledgments.** Thanks to E. Maderna and L. Rifford for the invitation to give the mini-course and to M.C. Arnaud for accepting to share the course with me. Discussions with S. Crovisier were important in the preparation of this notes (as well as in learning the material) and I thank him also for sharing with me some preliminary notes ([Cr_2]) he wrote about similar subjects. Thanks also to A. Passeggi for discussions and comments on the writting and G. Contreras for interesting input on the contributions of R. Mañé. Finally, I wish to dedicate this text to Ricardo Mañé, I did not know him personally, but his work has been a true inspiration for the whole mathematical community in Uruguay. The author was partially supported by Grupo CSIC 618.

2. Basics in differentiable ergodic theory

This section is devoted to present some basic results of ergodic theory which will be needed in the rest of the text. We shall restrict to the specific context we are interested in: M will denote a closed manifold and $f : M \to M$ a diffeomorphism of M. We refer the reader to [M_4] or [KH, Chapters 4 and 5] for a more complete account.

2.1. **Invariant and ergodic measures.** A probability measure μ in M will be said to be f-*invariant* if for every measurable set $A \subset M$ one has $\mu(f^{-1}(A)) = \mu(A)$.

We denote as $\mathcal{M}(f)$ the set of f-invariant probability measures. It is a standard fact that it is a compact convex subset of the space of measures with the weak-$*$ topology.

Exercise. Show that $\mathcal{M}(f)$ is non empty. (Hint: Consider the *empirical measures* $\mu_{n,x} = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ which are not invariant but as n grows, the defect of invariance decreases to 0).

There is a special important class of invariant measures which are called *ergodic*. A measure μ is called *ergodic* if every f-invariant set A verifies that either $\mu(A) = 0$ or $\mu(A^c) = 0$. We denote by $\mathcal{M}_{erg}(f)$ the subset of $\mathcal{M}(f)$ consisting of ergodic measures.

Exercise. Show that a f-invariant probability measure μ is ergodic if and only if for every f-invariant function φ one has that φ is constant μ-a.e.

One has that $\mathcal{M}_{erg}(f)$ is precisely the set of extremal points of $\mathcal{M}(f)$ (see [M$_4$]).

2.2. **Ergodic theorems.** We say that a sequence $\varphi_n : M \to \mathbb{R}$ is *subaditive* with respect to $f : M \to M$ if $\varphi_{n+m}(x) \leq \varphi_n(f^m(x)) + \varphi_m(x)$. The following result is by now classical:

Theorem 2.1 (Kingman)**.** *Let $f : M \to M$ preserving a measure μ and $\varphi_n : M \to M$ a subaditive sequence of functions such that $\varphi_1 \in L^1(\mu)$. Then, the sequence $\frac{1}{n}\varphi_n(x)$ converges μ-a.e. and in $L^1(\mu)$ to a f-invariant function $\tilde{\varphi} : M \to \mathbb{R}$ in $L^1(\mu)$ such that:*

$$\int \tilde{\varphi} d\mu = \inf_n \frac{1}{n} \int \varphi_n d\mu$$

A particularly concise proof of the pointwise convergence can be found in [AvB$_2$] (a proof which is in turn based on a proof of T. Kamae of Birkhoff's ergodic theorem which we partially reproduce below).

Given a function $\varphi : M \to \mathbb{R}$ we denote its n-th *Birkhoff sum* as:

$$S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(f^i(x))$$

It follows directly that the sequence $S_n\varphi$ is subaditive (in fact, additive) so the following is a direct consequence of Kingman's Theorem.

Theorem 2.2 (Birkhoff)**.** *Let $f : M \to M$ preserving a measure μ and $\varphi \in L^1(\mu)$. Then, the sequence $\frac{1}{n}S_n\varphi$ converges μ-ae and in $L^1(\mu)$ to a f-invariant function $\tilde{\varphi} \in L^1(\mu)$ and it follows that:*

$$\int \tilde{\varphi} d\mu = \int \varphi d\mu$$

In particular, if μ is ergodic then $\tilde{\varphi}(x) = \int \varphi d\mu$ for μ-ae x.

We give below a proof of the theorem for the particular (and important) case where φ is the characteristic function of a measurable subset $A \subset M$.

Proof of Theorem 2.2 for characteristic functions. (This should be skipped in a first reading.) Let $A \subset M$ be a measurable set and denote as $\varphi_n(x) = S_n\chi_A(x)$. Consider the following functions:

$$\underline{\tau}_A(x) = \liminf_n \frac{1}{n}\varphi_n(x) \quad ; \quad \overline{\tau}_A(x) = \limsup_n \frac{1}{n}\varphi_n(x)$$

Notice that one has that

$$\underline{\tau}_A(x) = \liminf_n \frac{1}{n}\varphi_n(x) = \liminf_n \frac{1}{n}(\chi_A(x) + \varphi_{n-1}(f(x))) = \underline{\tau}_A(f(x))$$

and therefore $\underline{\tau}_A$ is f-invariant. A symmetric argument shows that $\overline{\tau}_A$ is also f-invariant.

We want to show that for μ-almost every $x \in M$, one has that $\underline{\tau}_A(x) = \overline{\tau}_A(x) = \mu(A)$. Since one has obviously that $\underline{\tau}_A(x) \leq \overline{\tau}_A(x)$ for every x, it is enough to show that:

$$\int_M \underline{\tau}_A \geq \mu(A) \geq \int_M \overline{\tau}_A$$

The proofs are symmetric, so we shall only show that $\int_M \underline{\tau}_A \geq \mu(A)$.

Exercise. Use Fatou's lemma to show that $\int_M \underline{\tau}_A \leq \mu(A)$.

To show the inequality, fix $\varepsilon > 0$ and consider the sets

$$E_k = \{x \in M \;:\; \exists 1 \leq j \leq k \text{ such that } \frac{1}{j}\varphi_j(x) \leq \underline{\tau}_A(x) + \varepsilon\}$$

One has that $M = \bigcup_k E_k$ modulo a set of μ-measure zero.

We consider the functions $\psi_k : M \to [0,1]$ defined as follows: if $x \in E_k$ then $\psi_k(x) = \underline{\tau}_A(x) + \varepsilon$ and if $x \notin E_k^c$ then $\psi_k(x) = 1 + \varepsilon$. One has that the sequence ψ_k decreases to $\underline{\tau}_A(x) + \varepsilon$ as $k \to \infty$ (note that $\underline{\tau}_A(x) \leq 1$ for every $x \in M$).

By how we have defined ψ_k, whenever $n \geq k$ and $x \in E_k$, there is $j > 0$ such that $\varphi_n(x) = \varphi_{n-j}(f^j(x)) + \varphi_j(x)$ and such that $\varphi_j \leq j(\underline{\tau}_A(x) + \varepsilon)$. Since $\underline{\tau}_A(x)$ is invariant, one can write this as $\varphi_j(x) \leq \sum_{i=0}^{j-1} \psi_k(f^i(x))$. If $x \notin E_k$ it follows that $\varphi_n(x) = \varphi_{n-1}(f(x)) + \varphi_1(x)$ and $\varphi_1(x) \leq \psi_k(x)$ since $\varphi_1(x) \leq 1 < 1 + \varepsilon = \psi_k(x)$. Using this fact inductively, we know that for every x and $n \geq k$:

$$\varphi_n(x) = \sum_{i=0}^{n-1} \chi_A(f^i(x)) \leq k + \sum_{i=0}^{n-k-1} \psi_k(f^i(x)),$$

integrating in all M and using f-invariance of μ, one obtains:

$$\int_M \varphi_n \leq k + (n-k)\int_M \psi_k$$

Again by invariance of μ, one has that $\int_M \varphi_n(x) = n\mu(A)$, one deduces that:

$$n\mu(A) \leq k + (n-k)\int_M \psi_k$$

dividing by n and letting $n \to \infty$ one deduces:

$$\mu(A) \leq \int_M \psi_k$$

By monotone convergence one deduces that $\mu(A) \leq \int_M \underline{\tau}_A + \varepsilon$. Since ε was arbitrary, we deduce that $\mu(A) \geq \int_M \overline{\tau}_A$. Using a symmetric argument one obtains the other inequality and this concludes the proof of pointwise convergence of $\frac{1}{n}\varphi_n$. Since the functions are bounded by an integrable function, dominated convergence implies the L^1-convergence.

□

Exercise. Show that if a function $\phi : M \to \mathbb{R}$ verifies that $\phi \circ f - \phi$ is integrable, then $\lim_n \frac{1}{n}\phi(f^n(x)) = 0$ for μ-almost every $x \in M$.

2.3. **Periodic orbits and their splittings.** Let p be a fixed point of a C^1-diffeomorphism $f : M \to M$, that is, such that $f(p) = p$. It follows that $Df_p : T_pM \to T_pM$ induces a linear transformation of T_pM which is a finite dimensional linear space. As a consequence of the Jordan decomposition well known in linear algebra, one deduces that there exists a Df_p-invariant decomposition $T_pM = E_1 \oplus \ldots \oplus E_k$ associated to the[2] eigenvalues $\lambda_1, \ldots, \lambda_k$ of the linear transformation Df_p. One has that if a vector $v \in E_i \setminus \{0\}$ then the following is verified:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_p^n v\| = \log|\lambda_i|$$

Exercise. Let A be a matrix such that all eigenvalues have the same modulus equal to λ. Show that for every non-zero vector one has that $\lim_{n\to\pm\infty} \frac{1}{n}\log\|A^n v\| = \log|\lambda|$.

Once we have chosen to split the space in the eigenspaces corresponding to the eigenvalues of the same modulus, it is clear that the decomposition is unique.

A similar situation occurs when one has a periodic point p for f, i.e. $f^m(p) = p$ for some $m \geq 1$. Then, one obtains that p is a fixed point of f^m and therefore the splitting $T_pM = E_1(p) \oplus \ldots \oplus E_k(p)$ is Df_p^m-invariant and verifies that if $v \in E_i(p) \setminus \{0\}$:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_p^n v\| = \frac{1}{m}\log|\lambda_i|$$

If one considers $f^j(p)$ for some j, it is also a fixed point for f^m and therefore one can define a Df^m-invariant splitting $T_{f^j(p)}M = E_1(f^j(p)) \oplus \ldots \oplus E_k(f^j(p))$. Notice that k is independent of the iterate $f^j(p)$ since the linear transformations Df_p^m and $Df^m_{f^j(p)}$ are conjugate:

$$Df_p^m = Df^{-j}_{f^j(p)} Df^m_{f^j(p)} Df_p^j = (Df_p^j)^{-1} Df^m_{f^j(p)} Df_p^j$$

It follows from uniqueness that the relation: $Df^i_{f^j(p)} E_\ell(f^j(p)) = E_\ell(f^{i+j}(p))$ for every i,j and ℓ is verified.

Notice that eigenvalues can be defined regardless of the choice of a norm in T_pM since this is a well defined notion for vector spaces.

2.4. **Lyapunov exponents.** Invariant ergodic measures can be thought of as a generalization of periodic orbits.

Theorem 2.3 (Oseledets). *Let $f : M \to M$ be a C^1-diffeomorphism and μ an ergodic measure. Then, there exists $k \in \mathbb{Z}^+$, real numbers $\chi_1 < \chi_2 < \ldots < \chi_k$ and for x in a f-invariant full measure set $R^\mu(f)$ a splitting $T_xM = E_1(x) \oplus \ldots \oplus E_k(x)$ with the following properties:*

- *(**Measurability**) The functions $x \mapsto E_i(x)$ are measurable.*
- *(**Invariance**) $Df_x E_i(x) = E_i(f(x))$ for every $x \in R^\mu(f)$.*

[2]In the case where there are complex eigenvalues, we consider them in pairs $\lambda, \bar{\lambda}$ and the subspace corresponds to the real part of the sum of the spaces when considered as a complex linear transformation.

- *(***Lyapunov exponents***) For every $x \in R^\mu(f)$ and $v \in E_i(x) \setminus \{0\}$ one has*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Df_x^n v\| = \chi_i$$

- *(***Subexponential angles***) For every $x \in R^\mu(f)$ and vectors $v_i \in E_i(x)$ and $v_j \in E_j(x)$ one has that:*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \sin \measuredangle \left(\frac{Df_x^n v_i}{\|Df_x^n v_i\|}, \frac{Df_x^n v_j}{\|Df_x^n v_j\|} \right) = 0$$

Some explainations are in order:

2.4.1. *Lyapunov exponents.* The numbers χ_i appearing in the statement of Theorem 2.3 are usually called *Lyapunov exponents* of μ.

In general, for any diffeomorphism f a point $x \in M$ is called *regular* (or *Lyapunov regular*) if there exists a splitting $T_xM = E_1(x) \oplus \ldots \oplus E_{k(x)}(x)$ and numbers $\chi_1(x) < \chi_2(x) < \ldots < \chi_{k(x)}(x)$ such that for any vector $v \in E_i \setminus \{0\}$ one has that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Df_x^n v\| = \chi_i(x).$$

Exercise. Show that if $x \in M$ is a regular point and $v \in \bigoplus_{j=1}^{i} E_j(x) \setminus \bigoplus_{j=1}^{i-1} E_j(x)$ then

$$\lim_{n \to +\infty} \frac{1}{n} \log \|Df_x^n v\| = \chi_i(x).$$

In particular, every regular point verifies that every vector has a well defined Lyapunov exponent for the future (and the past). The bundles E_i are the ones on which both coincide.

The set of regular points $R(f)$ is f-invariant and Oseledets theorem implies that it has measure 1 for every f-invariant probability measure (one sometimes calls these sets *full measure sets*). It also holds that all the involved functions are measurable with respect to any invariant measure.

Notice that every periodic point has positive measure for an invariant measure (namely the one that gives equal weight to each point in the orbit) and therefore must be regular. Of course, one does not need Oseledets theorem to prove this, this follows exactly from the considerations in the previous section. Notice that if $f^n(p) = p$, then the Lyapunov exponents of p are the logarithms of the modulus of the eigenvalues of Df_p^n divided by n.

The *Pesin set* of f is the set of regular points for which all Lyapunov exponents are different from 0, that is, the set of points $x \in R(f)$ such that $\chi_i(x) \neq 0$ for all $1 \leq i \leq k(x)$. We shall see later why these points are relevant. A measure μ is called *(non-uniformly) hyperbolic* if all its Lyapunov exponents are non-zero: One should be careful with this name, the *non* applies to the uniformity and not to the hyperbolicity and it should be understood as "*not necessarily uniformly hyperbolic but still with a non-uniform form of hyperbolicity*".

For an ergodic (non-uniformly) hyperbolic measure μ for which one has Lyapunov exponents $\chi_1 < \ldots < \chi_i < 0 < \chi_{i+1} < \ldots < \chi_k$ one can group the bundles depending on the sign of the Lyapunov exponent. In this case, we denote $E^s(x) = E_1(x) \oplus \ldots \oplus E_i(x)$ and $E^u(x) = E_{i+1}(x) \oplus \ldots \oplus E_k(x)$. One has that if $v^s \in E^s(x) \setminus \{0\}$ and $v^u \in E^u(x)$ then:

$$\lim_{n\to\infty}\frac{1}{n}\log\|Df_x^n v^s\| < 0 < \lim_{n\to\infty}\frac{1}{n}\log\|Df_x^n v^u\|$$

So that vectors in $E^s(x)$ are the ones which are exponentially contracted in the future by Df and vectors in E^u are exponentially contracted in the past by Df.

2.4.2. *Angles and measurability.* We remark that, differently from the case of periodic orbits, the concept of norm and angle are essential in this setting as they provide a way to compare vectors which do not belong to the same vector space. However:

Exercise. The values of the Lyapunov exponents are independent of the choice of the Riemannian metric in TM.

The Riemannian metric also provides a way to compute angles between vectors and this is the sense one has to give to the last part of the statement of Theorem 2.3. It is possible to show that this last part is a consequence of the rest, but it is so important that it merits to appear explicitly in the statement.

Another relevant comment is about the notion of measurability of the functions $x \mapsto E_i(x)$. This should be understood in the following way: the arrow defines a function from M to the space of subspaces of TM. This can be thought of as a fiber bundle over M in the following way, for a given $j \leq d = \dim M$ one considers $G_j(M)$ to be the fiber bundle over M such that the fiber in each point is the Grasmannian space of T_xM of subspaces of dimension j. This is well known to have a manifold structure and provide a fiber bundle structure over M([3]). This gives a sense to measurable maps from M to some of these Grasmannian bundles, and since one does not a priori require that the bundles have constant dimension one can think of the function E_i to be a function from M to the union of all these bundles and then the measurability of the function makes sense as both the domain and the target of the function are topological spaces.

2.4.3. *Non-ergodic measures.* There is a statement for non-ergodic measures which is very much like the one we stated but for which the constants k and χ_i become functions of the points and some other parts become more tedious. Look [KH, Supplement] or [M_4, Chapter IV.10] for more information.

2.5. **Sketch of the proof of Oseledets theorem in dimension 2.** This section should be skipped in a first reading. For more details, see [AvB].

Consider $f : M \to M$ a C^1-diffeomorphism of a closed surface M. Let μ be an ergodic invariant measure.

Consider the sequence of functions $\varphi_n : M \to \mathbb{R}$ defined as $\varphi_n(x) = \log\|Df_x^n\|$. The chain rule together with the fact that the norm of a product of matrices is less than or equal to the product of their norms implies that the sequence $\varphi_n(x)$ is subaditive and thus Theorem 2.1 applies. Therefore, there exists $\chi_2 = \lim_n \frac{1}{n}\log\|Df_x^n\|$ for μ-almost every $x \in M$.

The same argument applied to f^{-1} implies the existence of

$$\chi_1 = -\lim_n \frac{1}{n}\log\|Df_x^{-n}\|$$

for μ-almost every $x \in M$. Since $\|Df^{-1}_{f(x)}\|^{-1} \leq \|Df_x\|$, one has that $\chi_2 \geq \chi_1$.

[3]For example, if $j = 1$ this is the projective bundle over M.

Exercise. Show that if $\chi = \chi_1 = \chi_2$ then for μ-almost every $x \in M$ and every $v \in T_xM \setminus \{0\}$ one has that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^n v\| = \chi.$$

We shall then concentrate on the case $\chi_2 > \chi_1$. The first remark is the following:

Exercise. Show that if $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear transformation verifying $\|A\| \neq \|A^{-1}\|^{-1}$, where $\|\cdot\|$ is associated to a given Euclidean metric. Then there exists orthogonal unit vectors $s \perp u$ such that $As \perp Au$ and

$$\|Au\| = \|A\| \quad ; \quad \|As\| = \|A^{-1}\|^{-1}.$$

The key to the proof is then to consider, for $x \in M$ such that the limits $\lim_n \frac{1}{n} \log \|Df_x^n\|$ and $\lim_n \frac{1}{n} \log \|Df_x^{-n}\|$ exist[4], the sequence of unit vectors s_n, u_n in T_xM defined such that $s_n \perp u_n$, $Df_x^n s_n \perp Df_x^n u_n$ and such that

$$\|Df_x^n u_n\| = \|Df_x^n\| \quad ; \quad \|Df_x^n s_n\| = \|(Df_x^n)^{-1}\|^{-1}.$$

One shows that the angle between s_n and s_{n+1} converges exponentially to 0 by using the fact that the limits above exist and the fact that $\|Df\|$ is uniformly bounded. Therefore there exists a limit $s = \lim s_n$ which verifies that

$$\lim_n \frac{1}{n} \log \|Df_x^n s\| = \chi_1.$$

It also follows that, for every unit vector v different from s one has that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n v\| = \chi_2.$$

The same argument for the past[5] gives the existence of $u \in T_xM$ such that

$$\lim_{n\to-\infty} \frac{1}{n} \log \|Df_x^n u\| = \chi_2.$$

One must then show that $s \neq u$. Then one can easily show that the angle between $s(f^n(x))$ and $u(f^n(x))$ decreases at subexponential rate with n because one has that for $v \neq w \in T_xM \setminus \{0\}$:

$$\|Df_x^{-1}\|^{-2} \leq \frac{\sin \measuredangle(Df_x v, Df_x w)}{\sin \measuredangle(v, w)} \leq \|Df_x\|^2$$

and therefore the function $x \mapsto \log \sin \measuredangle(s(f(x)), u(f(x))) - \log \sin \measuredangle(s(x), u(x))$ is bounded (and thus integrable). The details can be found in [AvB] in the more general case of linear $SL(2,\mathbb{R})$ cocycles.

2.6. **Pesin's reduction.** Oseledets Theorem 2.3 can be thought of as giving the "eigenvalues" of the derivative over an ergodic measure. We shall now present a result, due to Pesin, which can be then compared to "diagonalizing" the derivative over the measure (or putting it in Jordan form). Again, we treat a special case in dimension 2 for simplicity. See [KH, Supplement S] for more general versions.

[4]Notice that this is an f-invariant set.

[5]Notice that the limit of u_n exists and is orthogonal to s. However, this is not the vector we are interested in, since it might grow also for the past. We have to make a symmetric argument for f^{-1} to find the correct subspace.

Theorem 2.4 (Pesin's ν-reduction)**.** *Let $f : M \to M$ be a C^1-surface diffeomorphism and let μ be a ergodic measure with Lyapunov exponents $\chi_1 < \chi_2$. Then, for every $\nu > 0$ there exists a measurable function C_ν such that*[6] $C_\nu(x) \in GL(\mathbb{R}^2, T_xM)$ *and:*

- *(***Diagonalization***) There exists measurable functions functions $a_\nu : M \to (\exp(\chi_1 - \nu), \exp(\chi_1 + \nu))$ and $b_\nu : M \to (\exp(\chi_2 - \nu), \exp(\chi_2 + \nu))$ such that for μ-almost every point $x \in M$ one has that:*
$$C_\nu(f(x))^{-1} \cdot Df_x \cdot C_\nu(x) = \begin{pmatrix} a_\nu(x) & 0 \\ 0 & b_\nu(x) \end{pmatrix}$$
- *(***Subexponential decay of coordinate size:***) One has that for μ-almost every $x \in M$*
$$\lim_{n\to\pm\infty} \log(\|C_\nu(f^n(x))\| + \|(C_\nu(f^n(x)))^{-1}\|) = 0$$

The key part of the Theorem, which follows from the subexponential decay of the angles given by Oseledets theorem, is the fact that the norm of the matrices $C_\nu(f^n(x))$ and $(C_\nu(f^n(x)))^{-1}$ cannot grow to much along the orbit of generic points.

Sketch Let E_1 and E_2 be the measurable bundles given by Oseledets theorem associated to the exponents χ_1 and χ_2.

For a μ-generic point $x \in M$ one defines the vectors v_i as vectors in $E_i(x)$ of norm:

$$\left(\sum_{n\in\mathbb{Z}} \|Df_x^n|_{E_i(x)}\|^2 e^{-2n\chi_i} e^{-\nu|n|}\right)^{\frac{1}{2}}$$

The series converges for μ-almost every point thanks to the existence of Lyapunov exponents (and the extra term $e^{-\nu|n|}$). If one considers the linear transformation that sends the canonical base of $\mathbb{R}^2$ to v_1, v_2 one sees that the diagonalization hypothesis is easily verified.

Since $\|v_i\|$ is bounded from below, one has that the norm of $C(x)$ is uniformly bounded. On the other hand, the subexponential decay of of the angles given by Oseledets theorem as well as the fact that the Lyapunov exponents are the desired ones implies that the norm of $C(f^n(x))^{-1}$ is subexponential. See [KH, Theorem S.2.10] for more details.

□

3. Passing the information to the manifold

We shall restrict to dimension 2 for simplicity. So, in this section M will be a closed surface and $f : M \to M$ a diffeomorphism of M.

One can look at [KH, Section 6 and Supplement S] for more general statements. We remark that the proofs are quite similar in the higher dimensional context albeit more tedious in notation. The reader will notice that the calculations are already quite tedious in dimension 2.

[6] As above, one can define the function as a function from M to the bundle of linear maps from $\mathbb{R}^2$ to T_xM to make sense to the measurability. Alternatively, one can trivialize the tangent bundle of M up to a zero measure subset and then C_ν becomes a function from M to the space of 2×2 matrices.

The main point of this section is to show how one can recover the behavior seen at the level of the derivative in the dynamics in the manifold itself. The most detailed part will be the easiest one: the case of fixed points. Then, we shall try to explain how the other cases are simply complicated versions of the first one.

3.1. **Fixed points.** We shall work with $p \in M$ such that $f(p) = p$. Since we are in dimension two, we have the following possibilities:

- Both eigenvalues have modulus < 1 or both have modulus > 1.
- One eigenvalue has modulus < 1 and the other has modulus ≥ 1 or one eigenvalue has modulus > 1 and the other ≤ 1.
- Both eigenvalues have modulus 1.

The first case is the easiest to treat:

Exercise. Show that if both eigenvalues have modulus < 1 then p is a *sink*, i.e. there is a neighborhood U of p such that $f(\overline{U}) \subset U$ and for every $x \in U$ one has that $f^n(x) \to p$ exponentially fast. Symmetrically, if both eigenvalues have modulus > 1 the point p is a *source* (i.e. a sink for f^{-1}).

When both eigenvalues have modulus 1 less can be said. However, in dimension 2 there exist some results of topological flavor when the fixed point is isolated (see for example [LeR]).

When one non-zero Lyapunov exists, it is possible to reduce the dimension of the study via the following classical result:

Theorem 3.1 (Stable Manifold Theorem I). *Let p be a fixed point of a diffeomorphism $f : M \to M$ such that Df_p has one eigenvalue of modulus < 1 and the other has modulus ≥ 1. Then, there exists an embedded C^1 curve $\mathcal{W}^s_{loc}(p)$ with the following properties:*

- *(**Invariance**) One has $f(\mathcal{W}^s_{loc}(p)) \subset \mathcal{W}^s_{loc}(p)$*
- *(**Convergence**) For every $x \in \mathcal{W}^s_{loc}(p)$ one has that $d(f^n(x), p) \to 0$.*
- *(**Tangency**) The curve $\mathcal{W}^s_{loc}(p)$ is tangent to the subspace of T_pM corresponding to the eigenvalue of modulus < 1.*
- *(**Uniqueness**) If a point $x \in M$ satisfies that $d(f^n(x), p) \to 0$ exponentially fast, then there exists n_0 such that $f^{n_0}(x) \in \mathcal{W}^s_{loc}(p)$.*

The curve $\mathcal{W}^s_{loc}(p)$ is called the *local stable manifold* at p. One can consider the following:

$$\mathcal{W}^s(p) = \bigcup_{n>0} f^{-n}(\mathcal{W}^s_{loc}(p))$$

which we call the *stable manifold* of p.

Exercise. Show that $\mathcal{W}^s(p)$ is an injectively immersed curve diffeomorphic to $\mathbb{R}$. Give an example on which the manifold $\mathcal{W}^s(p)$ has finite length and an example where it has infinite length.

We shall give a quite detailed proof of Theorem 3.1 since many of the ideas will re-appear plenty of times later.

Proof. Consider a small neighborhood U of p and a chart $\varphi : U \to \mathbb{R}^2$ such that $\varphi(p) = 0$. By composing with a linear transformation, one can assume that $D\varphi$ sends the eigenspaces of Df_p to the axes of $\mathbb{R}^2$. Assume that the eigenvalue of modulus < 1 is sent to the horizontal axis.

Since there exists another neighborhood V of p such that $V \subset U$ and $f(V) \subset U$ we get that in $\varphi(V)$ one can define: $\hat{f} = \varphi \circ f \circ \varphi^{-1} : \varphi(V) \to \mathbb{R}^2$.

We can therefore write $\hat{f}$ in $\varphi(V)$ as:

$$\hat{f}(x,y) = (\lambda_1 x + \alpha(x,y), \lambda_2 y + \beta(x,y))$$

where $\lambda_1 < 1 \leq \lambda_2$ are the eigenvalues of Df_p and one has that $\alpha(0,0) = \beta(0,0) = \nabla\alpha(0,0) = \nabla\beta(0,0) = 0$. The functions α and β are C^1 on $\varphi(V)$ and therefore, given $\varepsilon > 0$ there exists $\delta > 0$ such that the C^1-size of α and β is smaller than ε in $B(0,\delta)$. Here the C^1 size is the maximum value between the images of the function and the norm of its partial derivatives. Notice that $D\hat{f}_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Consider a smooth bump function $\eta : \mathbb{R}^2 \to [0,1]$ with the following properties:

- $\eta(x,y) = 1$ if $\|(x,y)\| \leq \frac{\delta}{2}$.
- $\eta(x,y) = 0$ if $\|(x,y)\| \geq \delta$
- $\|\nabla\eta(x,y)\| \leq \frac{4}{\delta}$ for every (x,y).

We consider then the function $\bar{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $\bar{f} = \eta\hat{f} + (1-\eta)D\hat{f}_0$, i.e.:

$$\bar{f}(x,y) = \eta(x,y)\hat{f}(x,y) + (1-\eta(x,y))(\lambda_1 x, \lambda_2 y)$$

One can thus write:

$$\bar{f}(x,y) = (\lambda_1 x + \bar{\alpha}(x,y), \lambda_2 y + \bar{\beta}(x,y))$$

with $|\bar{\alpha}(x,y) - \bar{\alpha}(z,w)| \leq \bar{\varepsilon}\min\{\delta, \|(x,y)-(z,w)\|\}$ and $|\bar{\beta}(x,y) - \bar{\beta}(z,w)| \leq \bar{\varepsilon}\min\{\delta, \|(x,y)-(z,w)\|\}$. The value of $\bar{\varepsilon}$ can be chosen to be as small as desired by choosing δ, and ε correctly[7]. The advantage is that now we have a globally defined diffeomorphism of $\mathbb{R}^2$. Notice however that we can only say that the orbits by $\bar{f}$ represent orbits of $\hat{f}$ (or of f) while the point remains in $B(0,\frac{\delta}{2})$.

One can write $\bar{f}^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ as:

$$\bar{f}^{-1}(x,y) = (\lambda_1^{-1}x + \theta(x,y), \lambda_2^{-1}y + \vartheta(x,y))$$

again (maybe after re-choosing δ and ε) with the C^1-size of both θ and ϑ bounded by $\bar{\varepsilon}$.

Now, let us consider first the existence of a (unique) Lipschitz invariant curve for $\bar{f}$ tangent to the x-axis which is contracting.

Consider then the following complete metric space:

$$\mathrm{Lip}_1 = \{\varphi : \mathbb{R} \to \mathbb{R} \ : \ |\varphi(t) - \varphi(s)| \leq |t-s| \ , \ \forall t,s \ ; \ \varphi(0) = 0\}$$

endowed with the metric $d(\varphi, \varphi') = \sup_{t\neq 0 \in \mathbb{R}} \frac{|\varphi(t)-\varphi'(t)|}{t}$.

For a given $\varphi \in \mathrm{Lip}_1$ one can define a new function $\bar{f}_*\varphi$ as the function whose graph is the preimage by $\bar{f}$ of the graph of φ, i.e. $\operatorname{graph} \bar{f}_*\varphi = \bar{f}^{-1}(\operatorname{graph}\varphi)$.

Let us precise the construction of $\bar{f}_*\varphi$ a little further. Let $G_\varphi : \mathbb{R} \to \mathbb{R}$ the function defined by $G_\varphi(t) = \lambda_1 t + \bar{\alpha}(t, \varphi(t))$. One has:

Claim. *If $\bar{\varepsilon}$ is small enough, the function G_φ is an increasing homeomorphism of $\mathbb{R}$ which verifies* $(\lambda_1 - \sqrt{2}\bar{\varepsilon})|t-s| \leq |G_\varphi(t) - G_\varphi(s)| \leq (\lambda_1 + \sqrt{2}\bar{\varepsilon})|t-s|$.

[7]This is the well known fact that the C^1-topology is invariant under rescaling. Given $\bar{\varepsilon}$ there exists δ such that $\|\alpha(x,y)\|_{C^1} + \|\beta(x,y)\|_{C^1} \leq \frac{\bar{\varepsilon}\delta}{8}$ whenever $\|(x,y)\| \leq \delta$. Now, one has that the C^1-distance of $\bar{f}$ and $D\bar{f}_0$ is the C^1 size of $\eta(\bar{f} - D\bar{f}_0)$ which smaller than $\bar{\varepsilon}$ as desired.

Proof. Assume that $\sqrt{2}\bar\varepsilon < (1-\lambda_1)$. One computes:

$$|\lambda_1 t + \bar\alpha(t,\varphi(t)) - \lambda_1 s - \bar\alpha(s,\varphi(s))| \geq \lambda_1|t-s| - \sqrt{2}\bar\varepsilon|t-s| \geq (\lambda_1 - \sqrt{2}\bar\varepsilon)|t-s|$$

this follows from the fact that $|\bar\alpha(t,\varphi(t)) - \bar\alpha(s,\varphi(s))| \leq \bar\varepsilon\sqrt{|t-s|^2 + |\varphi(s)-\varphi(t)|^2}$ and that φ is 1-Lipschitz.

On the other hand, it is easy to see that $|\lambda_1 t + \bar\alpha(t,\varphi(t)) - \lambda_1 s - \bar\alpha(s,\varphi(s))| \leq (\lambda_1 + \sqrt{2}\bar\varepsilon)|t-s|$.

$\diamondsuit$

Then, the function $\bar f_*\varphi$ verifies $(t, \bar f_*\varphi(t)) = \bar f^{-1}(G_\varphi(t), \varphi(G_\varphi(t)))$ (see figure 1) and therefore:

$$\bar f_*\varphi(t) = \lambda_2^{-1}\varphi(G_\varphi(t)) + \vartheta(G_\varphi(t), \varphi(G_\varphi(t)))$$

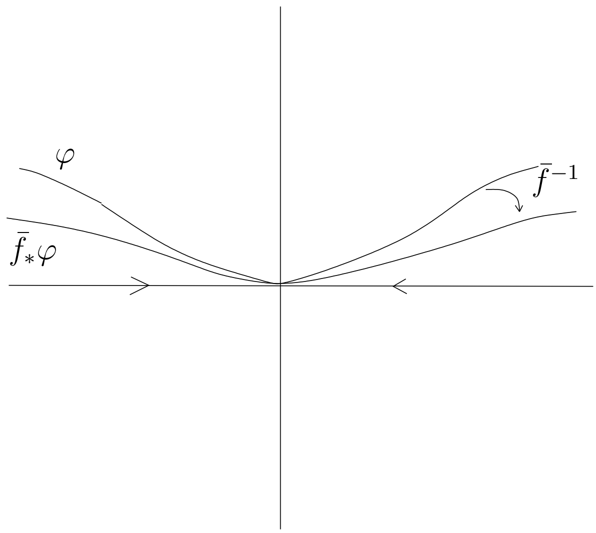

Figure 1. The graph transform of φ.

We have the following properties:

Claim. *If $\bar\varepsilon$ is small enough, for $\varphi \in \mathrm{Lip}_1$, the function $\bar f_*\varphi \in \mathrm{Lip}_1$.*

Proof. Intuitively, this follows directly from the fact that $D\bar f^{-1}$ contracts horizontal cones. Let us do the calculations (which should be skipped in a first reading). First notice that $\bar f_*\varphi(0) = 0$ from its definition.

Recall that for $t,s \in \mathbb{R}$ one has $|G_\varphi(t) - G_\varphi(s)| \leq (\lambda_1 + \sqrt{2}\bar\varepsilon)^{-1}|t-s|$

Given $t,s \in \mathbb{R}$ one has that:

$$\begin{aligned}
|\bar f_*\varphi(t)-\bar f_*\varphi(s)| &= |\lambda_2^{-1}\varphi(G_\varphi(t))+\vartheta(G_\varphi(t),\varphi(G_\varphi(t)))-\\
&\qquad -\lambda_2^{-1}\varphi(G_\varphi(s))+\vartheta(G_\varphi(s),\varphi(G_\varphi(s)))|\\
&\le \lambda_2^{-1}|\varphi(G_\varphi(t))-\varphi(G_\varphi(s))|+|\vartheta(t,G_\varphi(t))-\vartheta(s,\varphi(G_\varphi(s))|\\
&\le \lambda_2^{-1}|G_\varphi(t)-G_\varphi(s)|+\bar\varepsilon\,\|(t,\varphi(G_\varphi(t)))-(s,\varphi(G_\varphi(s)))\|\\
&\le \left(\lambda_2^{-1}(\lambda_1-\sqrt{2}\bar\varepsilon)+\sqrt{2}\bar\varepsilon\right)|t-s|
\end{aligned}$$

and if $\bar\varepsilon$ is small enough, one gets that $\lambda_2^{-1}(\lambda_1-\sqrt{2}\bar\varepsilon)+\sqrt{2}\bar\varepsilon<1$ as desired[8].

$\diamondsuit$

Claim. *For sufficiently small $\bar\varepsilon$, there exists $\gamma\in(0,1)$ such that if $\varphi,\varphi'\in \mathrm{Lip}_1$ then $d(\bar f_*\varphi,\bar f_*\varphi')\le\gamma d(\varphi,\varphi')$.*

Proof. Again, this is a consequence of the contraction of horizontal cones by $D\bar f^{-1}$. Let us perform the computations (the reader should skip them in a first reading).

$$\begin{aligned}
|\bar f_*\varphi(t)-\bar f_*\varphi'(t)| &= |\lambda_2^{-1}\varphi(G_\varphi(t))+\vartheta(G_\varphi(t),\varphi(G_\varphi(t)))-\\
&\qquad -\lambda_2^{-1}\varphi'(G_{\varphi'}(t))+\vartheta(G_{\varphi'}(t),\varphi'(G_{\varphi'}(t)))|\\
&\le \lambda_2^{-1}|\varphi(G_\varphi(t))-\varphi'(G_{\varphi'}(t))|+\\
&\qquad +|\vartheta(G_\varphi(t),\varphi(G_\varphi(t)))-\vartheta(G_{\varphi'}(t),\varphi'(G_{\varphi'}(t)))|\,.
\end{aligned}$$

Now, one has that

$$\begin{aligned}
|\varphi(G_\varphi(t))-\varphi'(G_{\varphi'}(t))| &\le |\varphi(G_\varphi(t))-\varphi'(G_\varphi(t))|+|\varphi'(G_\varphi(t))-\varphi'(G_{\varphi'}(t))|\\
&\le d(\varphi,\varphi')|G_\varphi(t)|+|G_\varphi(t)-G_{\varphi'}(t)|\\
&\le (\lambda_1+\sqrt{2}\bar\varepsilon)d(\varphi,\varphi')|t|+|\bar\alpha(t,\varphi(t))-\bar\alpha(t,\varphi'(t))|\\
&\le (\lambda_1+\sqrt{2}\bar\varepsilon)d(\varphi,\varphi')|t|+\sqrt{2}\bar\varepsilon d(\varphi,\varphi')|t|\\
&\qquad =(\lambda_1+2\sqrt{2}\bar\varepsilon)d(\varphi,\varphi')|t|
\end{aligned}$$

Moreover, one has that

$$\begin{aligned}
|\,\vartheta(G_\varphi(t),\varphi(G_\varphi(t))) \;-\;& \vartheta(G_{\varphi'}(t),\varphi'(G_{\varphi'}(t)))\,| \le\\
&\le |\vartheta(G_\varphi(t),\varphi(G_\varphi(t)))-\vartheta(G_\varphi(t),\varphi'(G_\varphi(t)))|+\\
&\qquad +|\vartheta(G_\varphi(t),\varphi'(G_\varphi(t)))-\vartheta(G_{\varphi'}(t),\varphi'(G_{\varphi'}(t)))|\\
&\le \bar\varepsilon d(\varphi,\varphi')|t|+\sqrt{2}\bar\varepsilon|G_\varphi(t)-G_{\varphi'}(t)|\\
&\le (2\sqrt{2}+1)\bar\varepsilon d(\varphi,\varphi')|t|
\end{aligned}$$

Putting all the estimates together it follows that if $\lambda_2^{-1}(\lambda_1+2\sqrt{2}\bar\varepsilon)+(2\sqrt{2}+1)\bar\varepsilon=\gamma<1$ one has the desired statement.

$\diamondsuit$

We deduce that there exists a unique function $\bar\varphi$ in Lip_1 whose graph is $\bar f$-invariant. We call $\mathcal{W}^s_{loc}$ to the restriction of the graph to $B(0,\frac{\delta}{2})$. The rest of the

[8]Notice that indeed, one does not need that $\lambda_2\ge 1$ but rather, it is enough that $\lambda_1<\lambda_2$ as long as one chooses $\bar\varepsilon$ correctly.

proof is devoted to showing that this graph (which is identified with a curve in M) verifies the conclusions of the theorem.

Invariance and convergence: This follows quite easily from the fact that if $|y| < t$ the map $t \mapsto \lambda_1 t + \bar{\alpha}(t, y)$ is contracting if $\bar{\varepsilon} < (1 - \lambda_1)$, therefore, since $\bar{\varphi} \in \text{Lip}_1$ one gets contraction for the map $t \mapsto \lambda_1 t + \bar{\alpha}(t, \bar{\varphi}(t))$ is contracting. This also implies that for every $(t, \bar{\varphi}(t))$ one has that $\bar{f}^n(t, \bar{\varphi}(t)) \to 0$ exponentially fast and that if $(t, \bar{\varphi}(t)) \in B(0, \frac{\delta}{2})$ (i.e. $(t, \bar{\varphi}(t)) \in \mathcal{W}^s_{loc}$) then[9] the same holds for $\bar{f}^n(t, \bar{\varphi}(t))$.

Smoothness: We must show that the curve $\mathcal{W}^s_{loc}$ is C^1 and tangent to the x-axis in $(0,0)$. To do so, notice that at each $t_0 \in \mathbb{R}$ one has that the set of accumulation points of

$$\frac{\bar{\varphi}(t) - \bar{\varphi}(t_0)}{t - t_0} \qquad \text{as} \quad t \to t_0$$

is an interval contained in $[-1, 1]$ because $\bar{\varphi} \in \text{Lip}_1$. This is equivalent to say that at each point $(t_0, \bar{\varphi}(t_0))$ the graph of $\bar{\varphi}$ is tangent to a cone of bounded width and transverse to the y-axis. The form of $\bar{f}$ implies that the angle of such a cone is contracted by an uniform amount by $D\bar{f}$. Using the fact that the graph of $\bar{\varphi}$ is $\bar{f}$-invariant, one deduces that the cones must degenerate at each point, or equivalently, the function $\bar{\varphi}$ is everywhere differentiable. A similar argument shows that these tangent spaces have to vary continuously with the point since otherwise one would obtain another invariant cone (by comparing the limits of different subsequences) by $D\bar{f}$ of positive width.

In $(0,0)$ it is clear that the unique direction transverse to the y-axis which is $D\bar{f}$-invariant is the x-axis and therefore the derivative of $\bar{\varphi}$ at 0 is 0 or equivalently, the curve $\mathcal{W}^s_{loc}$ is tangent to the x-axis at $(0,0)$ as desired.

Uniqueness: Assume that there is a point which converges exponentially fast to p for f. Then, one can construct a point $(t_0, s_0) \in B(0, \frac{\delta}{2})$ which converges exponentially fast to $(0,0)$ for $\bar{f}$. Since $\lambda_2 \geq 1$ one has that if $(t_n, s_n) = \bar{f}^n(t_0, s_0)$ then $\frac{s_n}{t_n}$ converges to zero since otherwise, the rate of convergence of (t_n, s_n) to zero is governed by λ_2 at first order[10]. Then, it is possible to construct a subfamily of Lip_1 consisting of functions such that $\varphi(t_n) = s_n$ for every n and one gets that it is a closed $\bar{f}_*$-invariant subset of Lip_1 and therefore, it contains the (unique) fixed point of the contraction. This proves the uniqueness.

□

Remark 3.2. Notice that the only place where we used that $\lambda_1 < 1$ is to show uniqueness (on the other hand, we used $\lambda_1 < \lambda_2$ everywhere). Otherwise, we would get a *locally invariant* curve which depends on the way we choose the extension (which is not canonical) and uniqueness only holds for points whose forward orbit remains in $B(0, \frac{\delta}{2})$. This is the content of the well known *center manifold* theorems ([Sh, HPS]).

Exercise (Chapter 5.III of [Sh])**.** Consider the time one map of the differential equation $\dot{x} = -x$ and $\dot{y} = y^2$ in $\mathbb{R}^2$. Show that $\mathcal{W}^s_{loc}(0,0)$ is the horizontal axis but

---

[9]The reader might be worried with the fact that the ball is round and therefore the contraction of the x-coordinate might not imply that the point remains in the ball. However, we notice that the intersection of a 1-Lipschitz graph through $(0,0)$ with a ball must be connected.

[10]Indeed, it is enough to show that $\frac{s_n}{t_n} \leq 1$. Notice that if $\lambda_2 > 1$ then the argument is simpler since points (t_0, s_0) such that $s_0 \geq t_0$ verify that its iterates by $\bar{f}$ leave $B(0, \frac{\delta}{2})$.

that uniqueness of the manifold tangent to the other direction is not ensured in the place where the dynamics tangent to the y-axis is contracting.

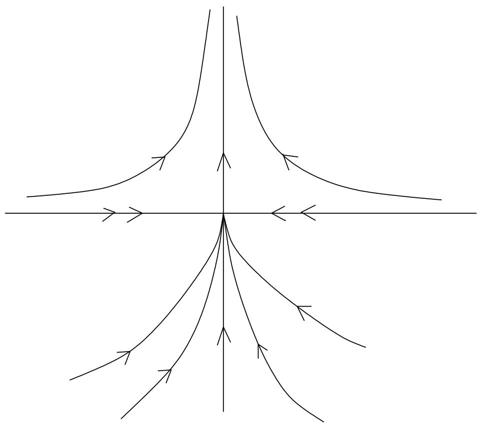

FIGURE 2. The flow of the equation $\dot{x} = -x$ and $\dot{y} = y^2$ in $\mathbb{R}^2$.

3.2. **The case of all Lyapunov exponents negative.** It is easy to pass from the information we gathered for fixed points to periodic points. The motivation from now on is to try to understand what kind of behavior is forced for general ergodic measures. It is natural to expect that zero-Lyapunov exponents will not provide much information, but when the measure is hyperbolic, one expects to obtain some information on the local dynamics for generic points of the measure.

This is an easier version of what follows, we shall see the first relatively easy consequence of a measure having non-zero Lyapunov exponents.

Theorem 3.3. *Let μ be an ergodic measure of a C^1-diffeomorphism f of a surface M such that both Lyapunov exponents are negative. Then, μ is supported in a periodic sink.*

Of course a symmetric statement holds for measures having all Lyapunov exponents positive where one obtains a periodic source applying the previous result to f^{-1}.

PROOF. One has that there exists $\chi < 0$ such that for μ-almost every $x \in M$ and every $v \in T_xM \setminus \{0\}$ one has that:

$$\limsup_n \frac{1}{n} \log \|Df_x^n v\| < \chi < 0.$$

Claim. *There exists $N_0 > 0$ such that for μ-almost every $x \in M$ and $N \geq N_0$ one has that*

$$\frac{1}{kN} \sum_{i=0}^{k-1} \log \|Df^N(f^{iN}(x))\| \to \hat{\chi}(x) \leq \chi$$

PROOF. We assume that μ is ergodic for f^N for every $N > 0$. Notice that it might be that it has (finitely) many ergodic components, the proof in this more general case is a little bit more tedious (see [AbBC, Lemma 8.4]).

The fact that all Lyapunov exponents are smaller than χ implies that

$$\frac{1}{n}\int \log \|Df^n\| d\mu \to \hat{\chi} \leq \chi$$

as $n \to \infty$. In particular, for sufficiently large N_0 one has that if $N \geq N_0$ then

$$\frac{1}{N}\int \log \|Df^N\| d\mu \leq \chi \,.$$

Now, the result follows from applying Birkhoff's theorem to the dynamics f^N and the function $x \mapsto \log \|Df^N(x)\|$.

$\diamondsuit$

Let us fix $\varepsilon \leq \frac{-\chi}{10}$, a value of $N \geq N_0$ as given by the previous claim and let $\Delta_f \geq \max_x \|Df(x)\|$.

There exists $\delta_0 > 0$ such that if $d(x,y) \leq \delta_0$ then for every vector $v \in T_yM$ one has that

$$\|Df^N(y)v\| \leq e^{N\varepsilon}\|Df^N(x)\|\|v\|$$

Let $R : M \to \mathbb{R}$ be defined as[11]:

$$R(x) = \max_{k\geq 0}\left\{ e^{-kN(\chi+\varepsilon)} \prod_{i=0}^{k-1} \|Df^N(f^{iN}(x))\| \right\} \geq 1$$

Notice that the previous claim implies that the value of $R(x)$ is well defined[12] on generic points with respect to μ since for sufficiently large k the value of $\prod_{i=0}^{k-1} \|Df^N(f^{iN}(x))\| \leq e^{kN(\chi+\varepsilon)}$.

Now consider $\delta_1 < \Delta_f^{-N}\delta_0$ and for μ-almost every $x \in M$ consider $\rho(x) = \frac{\delta_1}{R(x)}$. We have the following (compare with [AbBC, Lemma 8.10]):

Claim. *For μ-almost every $x \in M$ and $n \geq 0$ one has $f^n(B(x,\rho(x))) \subset B(x,\delta_0)$. Moreover, the diameter of $f^n(B(x,\rho(x)))$ converges to zero exponentially fast as $n \to \infty$.*

PROOF. Let us first prove that $f^n(B(x,\rho(x)) \subset B(x,\delta_0)$. Assume that this is the case for $k \leq n-1$. Consider $\ell \geq 0$ the largest integer for which $\ell N \leq n$. By induction and noticing that the derivative in $f^i(x)$ is approximately the same in points in $B(f^i(x),\delta_0)$ one shows that:

$$\operatorname{diam}(f^n(B(x,\rho(x)))) \leq \left(\Delta_f^N e^{\ell N\varepsilon} \prod_{i=0}^{\ell-1} \|Df^N(f^{iN}(x)\|\right)\rho(x) \leq \delta_0$$

One deduces using the definition of $R(x)$ that:

$$\operatorname{diam}(f^n(B(x,\rho(x)))) \leq \Delta_f^N e^{\ell N(\chi+2\varepsilon)} R(x)\frac{\delta_1}{R(x)} \leq e^{\ell N(\chi+2\varepsilon)}\Delta_f^N\delta_1 \leq \delta_0$$

But we have also established that

$$\operatorname{diam}(f^n(B(x,\rho(x))) \leq e^{\ell N(\chi+2\varepsilon)}\Delta_f^N\delta_1$$

---

[11] We make the convention that $\prod_{i=0}^{0} a_i = 1$.

[12] Indeed, standard arguments give that the sequence $R(f^n(x))$ is subexponential. See [AbBC, Lemma 8.7].

for every $n \geq 0$ which implies that the diameter goes to zero exponentially fast.

$\diamondsuit$

Consider a generic point x for μ, which is recurrent, i.e. there exists $n_j \to \infty$ such that $f^{n_j}(x) \in \Lambda_\varepsilon$ and $f^{n_j}(x) \to x$ and verifies the conditions of the previous claim. Such a point exists thanks to Poincare's recurrence theorem.

For large enough j, one has that $d(f^{n_j}(x), x) \ll \rho(x)$ and therefore

$$f^{n_j}(B(x, \rho(x))) \subset B(x, \rho(x))$$

and distances are contracted uniformly. This implies that $f^{n_j}|_{B(x,\rho(x))}$ has a unique (attracting) fixed point p and that $f^{kn_j}(y) \to p$ for every $y \in B(x, \rho(x))$. Since x was recurrent, this implies that $x = p$, which must be a periodic sink and this concludes the proof.

$\square$

Exercise. Prove Poincare's recurrence theorem (the statement used in the proof of Theorem 3.3) using Birkhoff's ergodic theorem.

3.3. **A result on sequences of diffeomorphisms.** We treat in this section a situation similar to the one we reduced in the fixed point case. Instead of dealing with a unique global diffeomorphism of $\mathbb{R}^2$ which is C^1-close to a linear transformation, we shall deal with a sequence of such maps and "notice" that we never really used the exact properties of the global diffeomorphism but instead we used the fact that the bounds were uniform. The reader can try to predict what purpose the result in this subsection will serve: one will consider charts around each point and extend the maps to global diffeomorphisms by using a bump function to glue the map with its derivative.

Let us introduce the context on which we shall work: A sequence $\{f_n\}_{n\in\mathbb{Z}}$ of diffeomorphisms of $\mathbb{R}^2$ is called a $(\lambda_1, \lambda_2, \varepsilon)$-*hyperbolic sequence of diffeomorphism* if it satisfies the following properties:

- $f_n(x, y) = (a_n x + \alpha_n(x, y), b_n y + \beta_n(x, y))$ where $0 < a_n < \lambda_1 < 1 < \lambda_2 < b_n$ and $\alpha_n(0, 0) = \beta_n(0, 0) = \nabla\alpha_n(0, 0) = \nabla\beta_n(0, 0) = 0$.
- The maps $\alpha_n : \mathbb{R}^2 \to \mathbb{R}$ and $\beta_n : \mathbb{R}^2 \to \mathbb{R}$ are C^1 and their C^1-distance to 0 is $< \varepsilon$. That is, for every $(x, y) \in \mathbb{R}^2$ one has that

$$|\alpha_n(x, y)|, \ \ |\beta_n(x, y)|, \ \ \|\nabla\alpha_n(x, y)\| \ \text{ and } \ \|\nabla\beta_n(x, y)\|$$

  are all smaller than ε.

The main result of this subsection is:

Theorem 3.4 (Stable Manifold Theorem for Hyperbolic Sequences)**.** *Given $\lambda_1 < 1 < \lambda_2$, there exists $\varepsilon > 0$ such that if $\{f_n\}_{n\in\mathbb{Z}}$ is a $(\lambda_1, \lambda_2, \varepsilon)$-hyperbolic sequence of diffeomorphisms, then, there exists a family of C^1 functions $\varphi_n : \mathbb{R} \to \mathbb{R}$ such that:*

- *(**Invariance**) The graphs are f_n-invariant, i.e. for every $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ there exists $s \in \mathbb{R}$ such that $f_n(t, \varphi_n(t)) = (s, \varphi_{n+1}(s))$.*
- *(**Convergence**) For every $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ one has that*

$$\lim_{m\to\infty} \|f_{n+m} \circ \ldots \circ f_n(t, \varphi_n(t))\| = 0\,.$$

- *(**Tangency**) The derivative $\varphi_n'(0) = 0$ for every $n \in \mathbb{Z}$.*
- *(**Uniqueness**) The family is the unique family with the first two properties.*

Indeed, the proof of this Theorem follows exactly the same lines as the proof we did in subsection 3.1. When looking at the proof of Theorem 3.1 one can identify two stages:

- First, one fixes a small chart around the fixed point where one can construct a global diffeomorphism of $\mathbb{R}^2$ which is C^1-close to a linear diagonal matrix with an eigenvalue smaller than one in the x-axis and larger than one in the y-axis.
- Then, one proves a result which is equivalent to Theorem 3.4 for a constant sequence $(\lambda_1, \lambda_2, \varepsilon)$-hyperbolic sequence of diffeomorphisms $f_n = \bar{f}$ for all $n \in \mathbb{Z}$ (to keep the notation of the proof of Theorem 3.1).

There is a minor difference on how to implement the proof. Instead of working with the space of Lipschitz functions (which has no longer much sense since the iterative process has to take place in "different" $\mathbb{R}^2$s), one has to work with sequences of Lipschitz functions. That is, one works with the space:

$$\mathrm{Lip}_1^{seq} = \{\{\varphi_n\}_n \ : \ \varphi_n(0) = 0 \ , \ |\varphi_n(t) - \varphi_n(s)| \leq |t - s|\}$$

endowed with the metric $d(\{\varphi_n\}, \{\varphi_n'\}) = \sup_{n \in \mathbb{Z}} d(\varphi_n, \varphi_n')$ which is also a complete metric space. One defines a *graph transform* of the form $\{f_n\}_*\{\varphi_n\} = \{\psi_n\}$ so that ψ_n is the function whose graph is the graph of $(f_n)^{-1}(\varphi_{n+1})$. The rest of the proof follows more or less verbatim as this graph transform preserves the space Lip_1^{seq} and contracts its metric giving a unique fixed point which will satisfy all the desired properties.

Exercise. Try to implement the same proof as in Theorem 3.1 to recover Theorem 3.4.

Theorem 3.4 is known as Hadamard-Perron's theorem. See [KH, Section 6.2] for more information and a complete proof in any dimension.

3.4. **Pesin charts.** The following result is the place where the $C^{1+\alpha}$ hypothesis appears in Pesin's theory. It allows to lift the dynamics to a subexponential neighborhood of a generic point for a hyperbolic measure μ and therefore obtain a hyperbolic sequence of diffeomorphisms. This allows to construct stable and unstable manifolds for those points using Theorems 2.4 and 3.4. By inspection of the proof one can see that the key place where the Hölder continuity of the derivative is used is to control the fact that angles can be very small (i.e. the norm of C_ν or C_ν^{-1} of Theorem 2.4 can be very large).

Theorem 3.5 (Pesin-Lyapunov Charts)**.** *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a closed surface M. Let μ be an ergodic measure with Lyapunov exponents $\chi_1 > \chi_2$. Then, for every $\rho_0 > 0$ and $\nu > 0$ there exists a measurable function $\rho : M \to (0, \rho_0)$ and a family of smooth charts $\xi_z : B(0, \rho(p)) \subset \mathbb{R}^2 \to M$ indexed in a full μ-measure set of $z \in M$ with the following properties:*

- *(**Lift of the dynamics:**) The map $\tilde{f}_z : B(0, \frac{\rho(z)}{\|Df\|}) \to B(0, \rho(f(z)))$ defined as $\tilde{f}_z = \xi_{f(z)}^{-1} \circ f \circ \xi_z$ is well defined and can be extended to a diffeomorphism $\hat{f}_z : \mathbb{R}^2 \to \mathbb{R}^2$ of the form:*
$$\hat{f}_z(x, y) = (a_z x + \alpha(x, y), b_z y + \beta(x, y))$$
where $\log a_z \in [\chi_2 - \nu, \chi_2 + \nu]$ and $\log b_z \in [\chi_1 - \nu, \chi_1 + \nu]$ and $\alpha(0, 0) = \beta(0, 0) = \nabla\alpha(0, 0) = \nabla\beta(0, 0) = 0$.

- (**Extension:**) *One can choose the extension* $\hat{f}_z$ *in such a way that the maps* $\alpha : \mathbb{R}^2 \to \mathbb{R}$ *and* $\beta : \mathbb{R}^2 \to \mathbb{R}$ *are* C^1 *and their* C^1*-distance to* 0 *is less than* ν*. That is, for every* $w \in \mathbb{R}^2$ *one has that* $|\alpha(w)|, |\beta(w)|, \|\nabla\alpha(w)\|$ *and* $\|\nabla\beta(w)\|$ *are all smaller than* ν*.*
- (**Subexponential decay of size:**) *The function* $\rho : M \to (0, \rho_0)$ *satisfies that* $\rho(f(z)) \in (e^{-\nu}\rho(z), e^{\nu}\rho(z))$ *and* $\lim_n \frac{1}{n} \log \rho(f^n(z)) = 0$*.*

PROOF. Let $\exp : TM \to M$ be the exponential mapping with respect to a given Riemannian metric. We know that $\exp_z : T_zM \to M$ verifies that $\exp_z(0) = z$ and $D(\exp_z)_0 = Id$. Using compactness of M we know that there exists $R_0 > 0$ such that $\exp_z : B(0, R_0) \to M$ is a diffeomorphism verifying that

$$\|(D\exp_z)_w\|, \|((D\exp_z)_w)^{-1}\|^{-1} \leq 2$$

for every $w \in B(0, R_0) \subset T_zM$.

Consider the linear change of coordinates $C_\nu(z) \in GL(\mathbb{R}^2, T_zM)$ given by 2.4 such that for μ-almost every point $z \in M$ one has that:

$$C_\nu(f(z))^{-1} \cdot Df_z \cdot C_\nu(z) = \begin{pmatrix} a_\nu(z) & 0 \\ 0 & b_\nu(z) \end{pmatrix}$$

and $a_\nu : M \to (\exp(\chi_1 - \nu), \exp(\chi_1 + \nu))$ and $b_\nu : M \to (\exp(\chi_2 - \nu), \exp(\chi_2 + \nu))$. One can choose $C_\nu(z)$ so that $\lim_{n \to \pm\infty} \log(\|C_\nu(f^n(z))\| + \|(C_\nu(f^n(z)))^{-1}\|) = 0$ for μ-almost every $z \in M$.

The function ξ_z will be the restriction of $\tilde{\xi}_z := (\exp_z \circ C_\nu(z)) : \mathbb{R}^2 \to M$ to a convenient neighborhood of 0.

First we shall define $\rho_1 : M \to (0, 1]$ to be a function verifying that for $z \in M$ the value $\rho_1(z)$ is the maximal value ≤ 1 such that:

- $C_\nu(z)(B(0, \rho_1(z))) \subset B(0, R_0)$ and
- $C_\nu(f(z))^{-1} \circ \exp_{f(z)}^{-1} \circ f \circ \exp_z \circ C_\nu(z)(B(0, \rho_1(z))) \subset B(0, R_0)$.

Technically, the function ρ_1 is only defined in points where C_ν is defined, but these form a full μ-measure set, so it is no problem for our purposes. In these points, the function is clearly positive and well defined. Moreover, the function $\tilde{\xi}_z$ is a diffeomorphism when restricted to $B(0, \rho_1(z))$ and we can therefore define:

$$\tilde{f}_z : B(0, \rho_1(z)) \to \mathbb{R}^2 \ , \quad \tilde{f}_z = \tilde{\xi}_{f(z)}^{-1} \circ f \circ \tilde{\xi}_z$$

The key difficulty is to obtain that the lift of f is C^1-close to its linear part $D(\tilde{f}_z)_0 = C_\nu(f(z))^{-1} \cdot Df_z \cdot C_\nu(z)$ (i.e. that the functions α and β are C^1-close to 0). It is for this that we shall restrict ρ further and use the $C^{1+\alpha}$ hypothesis.

We write $\tilde{f}_z = D(\tilde{f}_z)_0 + h_z$ where the function $h_z = (\alpha(z), \beta(z))$ and α and β verify $\alpha(0,0) = \beta(0,0) = \nabla\alpha(0,0) = \nabla\beta(0,0) = 0$.

In $B(0, R_0/\|Df_z\|)$ we can write $\exp_{f(z)}^{-1} \circ f \circ \exp_z = Df_z + g_z$ where g_z is $C^{1+\alpha}$ with similar constant as f (notice that exp is C^∞ and R_0 is chosen so that the derivative is controlled). So, there exists $c > 0$ such that for $w \in B(0, R_0/\|Df_z\|)$ one has:

$$\|(Dg_z)_w\| \leq c\|w\|^\alpha$$

Since $h_z = C_\nu(f(z))^{-1} \circ g_z \circ C_\nu(z)$, we have that

$$D(h_z)_w = D(C_\nu(f(z))^{-1} \circ g_z \circ C_\nu(z))_w = C_\nu(f(z))^{-1} \circ D(g_z)_{C_\nu(z)w}$$

so

$$\|D(h_z)_w\| \leq \|C_\nu(f(z))^{-1}\| \|D(g_z)_{C_\nu(z)w}\| \leq c\|C_\nu(f(z))^{-1}\| \|C_\nu(z)\|^\alpha \|w\|^\alpha$$

Notice that from the hypothesis on C_ν we know that

$$k(z) = c\|C_\nu(f(z))^{-1}\| \|C_\nu(z)\|^\alpha$$

has subexponential decay (i.e. $\lim_n \frac{1}{n} \log k(f^n(z)) = 0$) and if we choose $\rho_2 : M \to (0, \rho_0)$ small enough so that the norm of $\|D(h_z)_w\|$ is smaller than ν it is not hard to extend the functions $\tilde{f}_z$ to satisfy the extension property.

It remains to show that one can now choose $\rho : M \to (0, \rho_0)$ such that:

- $\rho(z) \leq \rho_2(z)$ for every $z \in M$,
- $\rho(f(z)) \in (e^{-\nu}\rho(z), e^\nu \rho(z))$
- $\lim_n \frac{1}{n} \log \rho(f^n(z)) = 0$.

The third condition follows immediately from the second. To construct ρ verifying the first two properties, it is enough to consider

$$\rho(z) := \inf_{n \in \mathbb{Z}} e^{\frac{\nu|n|}{4}} \rho_2(f^n(z))$$

which is well defined since $\lim_n \frac{1}{n} \log \rho_2(f^n(z)) = 0$ and verifies the desired properties. This concludes the proof of the Theorem.

□

Remark 3.6. By construction, one sees that there exists a measurable $K : M \to [1, \infty)$ such that if $w, w' \in B(0, \rho(z))$ then

$$d(\xi_z(w), \xi_z(w')) \leq \|w - w'\| \leq K(z) d(\xi_z(w), \xi_z(w'))$$

and such that $\lim_n \frac{1}{n} \log K(f^n(z)) = 0$.

One obtains the following result applying Theorems 3.4 and 3.5 (and Remark 3.6) which provides the so called Pesin's stable and unstable manifolds.

Theorem 3.7 (Pesin stable manifold theorem). *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a closed surface and μ a hyperbolic measure for f with Lyapunov exponents $\chi^s < 0 < \chi^u$. Then, there exists an f-invariant subset $R \subset M$ such that $\mu(R) = 1$ and:*

- *(**Existence:**) for every $x \in R$ there exists a C^1-curve $\mathcal{W}^s_{Pes}(x)$ centered at x and tangent to $E^s(x)$ with length $2\rho(x)$,*
- *(**Invariance:**) one has that $f(\mathcal{W}^s_{Pes}(x)) \subset \mathcal{W}^s_{Pes}(f(x))$,*
- *(**Convergence:**) for $y \in \mathcal{W}^s_{Pes}(x)$ one has that $\frac{1}{n}\log(d(f^n(x), f^n(y))) \to \chi^s$ for $n \to +\infty$,*
- *(**Uniqueness:**) if a point $y \in M$ verifies that $\frac{1}{n}\log(d(f^n(x), f^n(y))) \to \chi^s$ as $n \to +\infty$ then there exists n_y such that $f^{n_y}(y) \in \mathcal{W}^s_{Pes}(f^{n_y}(x))$,*
- *(**Size:**) the function $\rho : M \to (0, \rho_0)$ verifies that*

$$\rho(f(x)) \in (e^{-\nu}\rho(x), e^\nu \rho(x))$$

for $\nu \ll \min\{|\chi^s|, \chi^u\}$ and therefore $\lim \frac{1}{n} \log \rho(f^n(x)) = 0$.

3.5. **The $C^{1+\alpha}$ hypothesis.** In [BoCS] an example is constructed showing the importance of the Hölder continuity of the derivative in order to construct the stable and unstable manifolds for generic points with respect to a hyperbolic measure. In their example, the measure is hyperbolic but every point in the support of the measure verifies that their stable (resp. unstable) manifold is reduced to a point. We refer the reader to that paper to see the construction which in higher dimensions (≥ 3) gives also open sets of diffeomorphisms where C^1-generic diffeomorphisms in those open subsets have these pathological type of hyperbolic measures. We remark that their examples verify that the measures have zero entropy, and it is possible in principle that positive entropy allows to recover some of the Pesin theory in the C^1-context. We also refer the reader to [AbBC] for other contexts where Pesin theory holds for C^1-diffeomorphisms.

4. Entropy and horseshoes in the presence of hyperbolic measures

4.1. **Shadowing for hyperbolic sequences of diffeomorphisms.** Again, for simplicity, we shall restrict to the case of surface diffeomorphisms.

Consider a hyperbolic sequence of diffeomorphism $\{f_n : \mathbb{R}^2 \to \mathbb{R}^2\}_n$ as defined in section 3.3. It is not hard to see that for every $R > 0$ and $z \in \mathbb{R}^2 \setminus \{0\}$ there exists n such that either $f_n \circ \ldots \circ f_0(z) \notin B(0,R)$ or $f_{-n+1}^{-1} \circ \ldots \circ f_0^{-1}(z) \notin B(0,R)$. Indeed, the only points for which is necessary to consider "negative iterates" are the points lying in the stable manifold of 0.

Here we shall consider small perturbations of the diffeomorphisms f_n and try to show that the existence of a bounded orbit remains true. This is usually called *shadowing* (at least, its applications as we shall see in subsection 4.3).

Let $\{f_n\}_n$ be a $(\lambda_1, \lambda_2, \varepsilon)$-hyperbolic sequence of diffeomorphisms and let $\{v_n = (x_n, y_n)\}_n \subset \mathbb{R}^2$ be a sequence of vectors. We consider the following sequence of diffeomorphism $\{f_n^v\}_n$ defined as:

$$\begin{aligned} f_n^v(x,y) &= (a_n x + \alpha_n(x,y), b_n y + \beta_n(x,y)) + (x_n, y_n) \\ &= (a_n x + x_n + \alpha_n(x,y), b_n y + y_n + \beta_n(x,y)) \end{aligned}$$

We say that a sequence $\{z_n\} \subset \mathbb{R}^2$ is an *orbit* of f_n^v if one has that $f_n^v(z_n) = z_{n+1}$ for every $n \in \mathbb{Z}$. An orbit $\{z_n\}_n$ is *bounded* if $\sup_{n\in\mathbb{Z}} \|z_n\| < \infty$.

Theorem 4.1 (Exponential Shadowing for hyperbolic sequences)**.** *Let f_n be a $(\lambda_1, \lambda_2, \varepsilon)$-hyperbolic sequence of diffeomorphisms (with $0 < \lambda_1 < 1 < \lambda_2$ and $\varepsilon \ll \min\{\lambda_2 - 1, 1 - \lambda_1\}$). Then, there exists $R_0 := R_0(\lambda, \mu, \varepsilon)$ such that for every $\delta > 0$ one has that if $\{v_n\}_n \subset \mathbb{R}^2$ is a sequence of vectors satisfying $\sup_n \|v_n\| \leq \delta$ then there exists a unique bounded orbit $\{z_n\}_n$ of $\{f_n^v\}_n$ which verifies the following:*

- $\sup_{n\in\mathbb{Z}} \|z_n\| \leq R_0\delta$ *and,*
- *the orbit $\{z_n\}_n$ is* uniformly hyperbolic*, that is, one has that for every $m \in \mathbb{Z}$ there exist subspaces E^s_m and E^u_m in $T_{z_m}\mathbb{R}^2$ such that $Df^v_m E^\sigma_m = E^\sigma_{m+1}$ (for $\sigma = s, u$) such that for every $n \geq 1$ one has*

$$\|D((f^v_m)^n)_{z_m} E^s_m\| \leq (\lambda_1 + 5\varepsilon)^n \ , \ \|D((f^v_m)^n)_{z_m} E^u_m\| \geq (\lambda_2 - 5\varepsilon)^n$$

Moreover, if $\{\tilde{f}_n\}_n$ is another $(\lambda_1, \lambda_2, \varepsilon)$-hyperbolic sequence of diffeomorphisms and $\{\tilde{v}_n\}_n$ another sequence of vectors such that $\sup_n \|\tilde{v}_n\| \leq \delta$ verifying that $\tilde{f}_k = f_k$

and $\tilde{v}_k = v_k$ *for every* $-M \le k \le M$ *then one has that if* $\{\tilde{z}_n\}_n$ *is the bounded orbit of* $\{\tilde{f}_n^{\tilde{v}}\}$ *then*

$$\|z_0 - \tilde{z}_0\| \le R_0\delta((\lambda_1 + 5\varepsilon)^M + (\lambda_2 - 5\varepsilon)^{-M})$$

Remark 4.2. It follows from uniqueness that if for some $m \ge 1$ one has $f_{n+m}^v = f_n^v$ for every $n \in \mathbb{Z}$ then $z_{n+m} = z_n$ for every $n \in \mathbb{Z}$.

PROOF. Consider R_0 large enough so that if one considers the square Q of side $2\delta R_0$ around $(0,0)$ one has that $f_n^v(Q)$ is a rectangle which traverses Q (see figure 3). It is clear that this can be done and the value of R_0 is independent of δ.

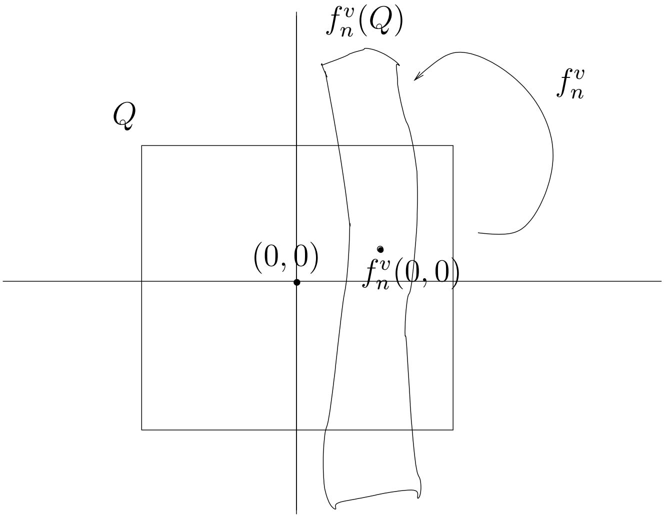

FIGURE 3. The image of Q by f_n^v.

As in the proof of Theorem 3.1 the square Q can be foliated by horizontal and vertical Lipschitz curves which allow to define a width of $f_n^v(Q)$. This width is contracted by a factor of $\lambda_1 + \varepsilon$. Moreover, the same argument for $(f_n^v)^{-1}$ implies that the height of $(f_n^v)^{-1}(Q) \cap Q$ is contracted by $(\lambda_2 - \varepsilon)^{-1}$. By an inductive argument one can show the existence of the desired orbit $\{z_n\}_n$ whose orbit stays always in Q (and therefore $\sup_n \|z_n\| \le R_0 d$). Moreover, its localization depends on the intersection of the iterates of the square, so the precision on which we know the location of z_n is exponential in M if we know the form of f_k^v for $-M \le k \le M$.

To show uniform hyperbolicity of the orbit $\{z_n\}$ it is enough to make a cone-field argument which is similar to the one that it is possible to make to construct stable and unstable manifolds for the orbit $\{z_n\}_n$ as in Theorem 3.4.

Finally, to show uniqueness, consider two different orbits $\{w_n\}_n$ and $\{w'_n\}_n$. If w_0 differs from w'_0 in the second coordinate more than in the first, it is not hard to see using the form of f_n that $\|w_n - w'_n\| \to \infty$ as $n \to \infty$. Similarly, if the first coordinate differs more than the second, then $\|w_n - w'_n\| \to \infty$ as $n \to -\infty$. This implies that $\sup_n \|w_n - w'_n\| = \infty$ so that only one orbit can be bounded.

□

4.2. **Metric entropy and Lyapunov exponents.** There exists an important relation between the entropy of an ergodic measure and its positive Lyapunov exponents. In a nutshell, entropy measures the exponential growth of the number of different segments of orbits of length n at a given precision as n goes to infinity. Lyapunov exponents measure the exponential speed at which points get separated.

It is therefore natural to expect that the entropy of a measure is bounded from above by the sum of its positive Lyapunov exponents (this is usually called Ruelle's inequality). Also, one expects that if a measure has positive Lyapunov exponents and the measure can "see" the unstable manifolds, then its entropy will be positive (this is also a well known general principle which can be attributed among others to Pesin and Ledrappier-Young). We shall only briefly review a small part of this rich theory and we refer the reader to [BaP] and references therein for a more complete account on this theory.

Before we define entropy of a measure and topological entropy we need some preliminaires. Let $f : M \to M$ be a diffeomorphism of a closed manifold M endowed with a distance d. We consider the *dynamical* or *Bowen balls* defined as:

$$B_n(x,\varepsilon) = \{y \in M \ : \ d(f^j(x), f^j(y)) \leq \varepsilon \ , \ 0 \leq j \leq n\}$$

One defines the *topological entropy* $h_{top}(f)$ of f as:

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_f(n,\varepsilon)$$

where $N_f(n,\varepsilon)$ is the smallest number $k > 0$ such that there exist points $x_1, \ldots, x_k$ verifying that $M = \bigcup_{i=1}^k B_n(x_i,\varepsilon)$.

For an ergodic f-invariant measure μ one defines[13] the *entropy* $h_\mu(f)$ as:

$$h_\mu(f) = -\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mu(B_n(x,\varepsilon)) \ , \quad \text{for } \mu\text{-almost every } x \in M$$

One can check that all the involved limits are well defined, etc (see [KH] or [M_4]).

It is a well known fact, known as the *Variational Principle* (see [M_4] for a proof) that the topological entropy is the supremum of the values of the entropies of the ergodic measures invariant under f, i.e.:

$$h_{top}(f) = \sup_{\mu \ ergodic} h_\mu(f)$$

This will be used in the two senses:

- if one knows that a diffeomorphism has positive topological entropy (for example, by knowing that the action in the first homology group is hyperbolic) then there exist measures with positive entropy,
- if there exists a measure with positive entropy, then the topological entropy is positive.

Entropy of a measure is related to Lyapunov exponents via the following results which we state for diffeomorphisms of surfaces for simplicity. See [BaP] for more general statements and proofs.

[13] This definition is due to Brin and Katok.

Theorem 4.3 (Ruelle's inequality)**.** *Let $f : M \to M$ be a C^1-diffeomorphism of a surface M. For an ergodic f-invariant measure μ one has that if $h_\mu(f) > 0$ then μ is hyperbolic with exponents $\chi^s < 0 < \chi^u$ and moreover:*

$$h_\mu(f) \leq \min\{|\chi^s|, \chi^u\}.$$

In general, the inequality can be strict. For example, if μ is a Dirac delta measure on a hyperbolic fixed saddle p then it is ergodic, invariant and clearly hyperbolic. On the other hand, it is easy to see that $h_\mu(f) = 0$ since $\mu(B_n(p, \varepsilon)) = 1$ for every n and ε.

To obtain entropy of a measure one needs that the measure "sees" the expansion. This can be formulated in the following form for surfaces (again, this is far from being optimal, see [BaP] for more general statements). The following result will follow from Katok's theorem which we shall review in the next section, but it admits more quantitative versions (which depend on desintegration of measures along a lamination and that is why we refer the interested reader to read this elsewhere, for example [BaP]).

Theorem 4.4. *Let μ be a hyperbolic measure of a $C^{1+\alpha}$ diffeomorphism whose support is not finite. Then $h_\mu(f) > 0$.*

There is however one case where the desintegration of the measure can be excluded from the statement and which is important in some contexts. We recall that a diffeomorphism is conservative if it preserves a volume form *vol*. In general, it is too restrictive to assume that *vol* is ergodic, so, in general, the Lyapunov exponents of *vol* are f-invariant functions instead of constants.

Theorem 4.5 (Pesin's entropy formula)**.** *Let $f : M \to M$ be a conservative $C^{1+\alpha}$-diffeomorphism of a surface M. Then one has that*

$$h_{vol}(f) = \int \chi^u dvol = -\int \chi^s dvol.$$

See [M_2] for a hands on proof which does not rely (explicitely) on the absolute continuity of the unstable lamination of the measure. In particular, this proof is one of the first instances where the use of the subexponential size of Pesin charts is used to study the dynamics without the need to construct the invariant manifolds first.

4.3. **Katok's theorem on the existence of horseshoes.** In this subsection we shall explain a stronger version of Theorem 4.4. It is by now a classical result due to Katok (see [KH, Supplement], [Ge] or the appendix of [AvCW] for more general versions and some improvements) that the existence of a hyperbolic measure which is not periodic implies the existence of horseshoes in the $C^{1+\alpha}$ context.

The following is the precise statement in the surface case:

Theorem 4.6 (Katok)**.** *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a closed surface M and μ an ergodic f-invariant measure whose support is not finite. Then, $h_\mu(f) > 0$ and for every $\varepsilon > 0$ there exists a compact f-invariant subset Λ contained in the support of μ such that:*

- *the set Λ is* uniformly hyperbolic*,*
- *the topological entropy $h_{top}(f|_\Lambda)$ of f restricted to Λ is close to $h_\mu(f)$, i.e.*

$$h_{top}(f|_\Lambda) > h_\mu(f) - \varepsilon$$

For the definition of uniform hyperbolicity we refer the reader to section 5. In a uniformly hyperbolic set with positive topological entropy there are infinitely many hyperbolic periodic orbits and it is not hard to show the existence of a *transverse homoclinic intersection* which is well known to produce a *horseshoe* (regardless of the definition of horseshoe that we have not given). We refer the reader to [KH, Chapter 6.5] for a more complete account. We shall explain now the main ingredients of the proof of Theorem 4.6.

The key point is to work on what are sometimes called *Pesin blocks* or *uniformity blocks*. These are subsets on which the constants are uniform (the choice of what constants are chosen to be uniform vary in the literature). For example, in this case, one can consider, for a given $K > 0$ the set Λ_K of points $x \in M$ such that the value of $\rho(x) \geq \frac{1}{K}$ of Theorem 3.5 as well as $\|C_\nu(x)\| + \|C_\nu(x)^{-1}\|^{-1} \leq K$ of 2.4. By reducing Λ_K up to an arbitrarily small measure subset, one can assume that Λ_K is compact. Moreover, for given $\varepsilon > 0$ there exists K such that Λ_K verifies that $\mu(\Lambda_K) \geq 1 - \varepsilon$. Also, it is a standard fact that one can assume that all the involved functions (ρ, C_ν, etc) are continuous on Λ_K (this is the classical Luisin's theorem).

If the set Λ_K were f-invariant this would conclude since it can be chosen as to have entropy as close to $h_\mu(f)$ as one desires. However, in general there is no reason to expect that Λ_K will be f-invariant.

One proceeds then as follows: one considers first a finite covering of the support of μ by Pesin charts which has a Lebesgue number and covers Λ_K. In Λ_K one has uniform charts where the constants of C_ν are bounded, and there remains a finite number of transition charts where points go when the iterates do not belong to Λ_K. In such a way one can construct a large number of "adapted orbits" of f which "see" almost all the entropy of μ and return to Λ_K quite frequently.

This allows to construct enough periodic pseudo-orbits which will be shadowed by periodic orbits of f which are uniformly hyperbolic because they belong to Λ_K. The way to "shadow" these orbits is using Theorem 4.1 by looking at orbits that remain always in the places where the lift of the dynamics given by Theorem 3.5 coincides with f. This step is the most delicate since to ensure that the orbits remain in the Pesin charts at all times one has to carefully choose the orbit one wishes to shadow in order to control its orbit. See [KH, Lemma S.4.10] for more details.

The fact that the periodic orbits one construct are hyperbolic and sufficiently close to each other allows one to show that they are all homoclinically related and therefore belong to the same f-invariant compact subset Λ which is transitive and uniformly hyperbolic. The entropy is as close to $h_\mu(f)$ as desired.

Recently, in [Sar], these arguments have been improved to construct sets which see *all* the entropy of μ. The idea is consider a Pesin chart at each point and consider a countable subcovering. Then, one considers the possible itineraries that orbits make through those charts and constructs a countable Markov partition with similar ideas as those of Bowen for constructing Markov partitions. The sets constructed in [Sar] cease to be uniformly hyperbolic but enjoy coding properties for which much information is known. See [Sar] for more details on this and the previous construction.

4.4. **When is a measure hyperbolic?** Clearly, because of Ruelle's inequality (Theorem 4.3) positive entropy is a sufficient condition for a measure to be hyperbolic in dimension 2. There are some times where one does not have enough information on the invariant measure in order to compute its entropy. It is therefore important to have other methods to guarantee existence of positive Lyapunov exponents. Here it is important to remark that in some (very important) applications one deals with non-ergodic (or a priori non-ergodic) measures where positive entropy only guarantees some ergodic component to have positive Lyapunov exponents. I would like to mention three known methods for establishing hyperbolicity of a measure.

The first one has been developed independently by Lewowicz and Wojkowski (see [Pot$_3$, Section 2.1] and references therein) and it is the method of measurable cone-fields or quadratic forms. This has been quite useful to establish hyperbolicity (and more recently the Bernoulli property) to a large class of billiards (see [DeM]).

The second is also related to cone-fields but it deals more with the notion of *critical points* and extends the ideas which were developed in the setting of one-dimensional dynamics. This was done famously by Benedicks-Carlesson ([BeC]) to study the parameters for which the Hénon family admits non-uniformly hyperbolic attractors and has been used largely since then (see [Ber$_2$] for improvements of that result as well as a panorama of the works related to this).

More recently, in the setting of conservative twist maps, Arnaud has developed some techniques to compute Lyapunov exponents for such maps and discovered some interesting relations of these with the shape of the so called Aubry-Mather set and Green bundles. Explaining this is possibly the objective of Marie-Claude's minicourse, but we also refer to her lecture notes [Arn] (and references therein) for more information.

5. Uniform estimates

In this section we shall briefly present the concepts of dominated splitting, partial hyperbolicity, normal hyperbolicity, etc. Essentially, one can think these concepts as uniform versions of the hyperbolicity of measures: If a compact set admits a continuous splitting of the tangent bundle such that for every measure supported in the compact set, there is a positive gap between the Lyapunov exponents along each of the bundles, then, the compact set is said to admit a dominated splitting. Under this conditions, it is no longer needed to have control on the modulus of continuity of the derivative in order to perform the graph-transform arguments. Moreover, under some assumptions of hyperbolicity, one obtains results of persistence of invariant manifolds which are quite useful in many applications; particularly (in view of the interests of this conference) we mention the use of normally hyperbolic cylinders in the recent proofs of Arnold diffusion ([BeKZ, GK, KZ] and references therein).

M will denote a d-dimensional manifold and $f : M \to M$ a C^1-diffeomorphism.

5.1. **Dominated splittings.** Let $\Lambda \subset M$ be a compact f-invariant set. We say that it admits a *dominated splitting of index i* if there is a continuous splitting $T_\Lambda M = E \oplus F$ (i.e. for every $x \in \Lambda$ one has $T_x M = E(x) \oplus F(x)$ and E and F are continuous functions) such that the bundle $E(x)$ has dimension i and verifies the following properties:

- (**Invariance**) The bundles are Df-invariant, that is, for every $x \in \Lambda$ one has $Df_x(E(x)) = E(f(x))$ and $Df_x(F(x)) = F(f(x))$.
- (**Domination**) There exists $N > 0$ such that for every $x \in \Lambda$ and vectors $v_E \in E(x) \setminus \{0\}$ and $v_F \in F(x) \setminus \{0\}$ one has that:

$$\frac{\|Df_x^N v_E\|}{\|v_E\|} < \frac{\|Df_x^N v_F\|}{\|v_F\|}$$

It follows from compactness that there exists $\lambda \in (0,1)$ such that

$$\frac{\|Df_x^N v_E\|}{\|v_E\|} < \lambda \frac{\|Df_x^N v_F\|}{\|v_F\|}$$

It is possible to choose an *adapted metric* for which the value of N is equal to 1 (see [Gou]). The fact that the splitting is *dominated* is independent of the choice of the Riemannian metric.

Exercise. Show that a continuous Df-invariant decomposition $T_\Lambda M = E \oplus F$ is dominated if and only if there exists $\nu > 0$ such that for every ergodic measure $\mu \in \mathcal{M}_{erg}(f)$ supported on Λ one has that the largest Lyapunov exponent $\chi_E^+(\mu)$ of μ along E and the smallest Lyapunov exponent $\chi_F^-(\mu)$ of μ along F verify:

$$\chi_E^+(\mu) \leq \chi_F^-(\mu) - \nu.$$

In particular, show that if the splitting is dominated, then the Oseledets splitting respects (and refines) the splitting $E \oplus F$.

It is not hard to show that when there is a dominated splitting on a subset Λ, the angle between the subbundles of the domination is uniformly bounded from below, this follows directly by continuity of the bundles and compactness of Λ. We remark here that the continuity of the bundles is not essential in the definition of domination and it follows from the rest of the properties (see [BoDV, Appendix B]).

A key property of dominated splitting is that it is *robust*:

Proposition 5.1. *Let $\Lambda \subset M$ be a compact set admitting a dominated splitting of the form $T_\Lambda M = E \oplus F$ for a diffeomorphism $f : M \to M$ of class C^1. Then, there exists a compact neighborhood U of Λ in M and a neighborhood $\mathcal{U}$ of f in the C^1-topology such that every $g \in \mathcal{U}$ verifies that the set $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ admits a dominated splitting $T_{\Lambda_g} M = E_g \oplus F_g$ with* $\dim E_g = \dim E$.

We shall not prove this result since it is not in the spirit of this notes, the proof is not hard, see for example [BoDV, Appendix B].

If Df preserves a continuous subbundle $E \subset T_\Lambda M$ we say that E is *uniformly contracted* (resp. *uniformly expanded*) if there exists $N > 0$ such that for every $x \in \Lambda$ and every unit vector $v \in E(x)$ one has that

$$\|Df_x^N v\| \leq \frac{1}{2} \quad (\text{resp.} \quad \geq 2)$$

Exercise. Show that a continuous Df-invariant subbundle $E \subset T_\Lambda M$ is uniformly contracted if and only if every ergodic measure μ supported on Λ has all Lyapunov exponents corresponding to vectors in E negative.

5.2. **Plaque families.** When one has a dominated splitting on a compact subset $\Lambda \subset M$, as we mentioned, one can assume that for every f-invariant ergodic measure verifies that its Oseledets splitting respects the splitting given by the domination. So, in a sense, this means that when considering linear change of coordinates which make the bundles orthogonal these changes of coordinates become uniformly bounded. It is in a sense as if the norm of the maps C_ν and C_ν^{-1} of Theorem 2.4 are uniformly bounded. This is not exactly true since the norm of C_ν and C_ν^{-1} also depend on how quickly the derivative starts behaving as its "limit behaviour".

Let us state another result ([HPS, Theorem 5.5]) which is still in the spirit of the graph transform argument. We shall only sketch its proof. We denote as $\mathbb{D}^k$ to the k-dimensional disk of unit radius in $\mathbb{R}^k$ and $\mathrm{Emb}^1(\mathbb{D}^k, M)$ to the space of C^1-embeddings of $\mathbb{D}^k$ into M. We denote as $\mathbb{D}^k_r \subset \mathbb{D}^k$ to the disk of radius $r \leq 1$

Theorem 5.2 (Plaque Families). *Let $f : M \to M$ be a C^1-diffeomorphism and $\Lambda \subset M$ a compact f-invariant subset admitting a dominated splitting of the form $T_\Lambda M = E \oplus F$. Then, there exists a continuous[14] family $\mathcal{D}_E : \Lambda \to \mathrm{Emb}^1(\mathbb{D}^{\dim E}, M)$ with the following properties:*

- *(**Tangency:**) for every $x \in \Lambda$ one has that $\mathcal{D}_E(x)(0) = x$ and the image of $\mathcal{D}_E(x)$ is tangent to $E(x)$ at x.*
- *(**Local invariance:**) there exists $r_0 < 1$ such that for every $x \in \Lambda$ one has that $f(\mathcal{D}_E(x)(\mathbb{D}^{\dim E}_{r_0})) \subset \mathcal{D}_E(f(x))(\mathbb{D}^{\dim E})$.*

Sketch Using continuity of the bundles one can choose[15] a continuous linear change of coordinates $C(x) : \mathbb{R}^d \to T_xM$ (recall that $d = \dim M$) such that $C(x)(\mathbb{R}^{\dim E} \times \{0\}^{\dim F}) = E(x)$ and $C(x)(\{0\}^{\dim E} \times \mathbb{R}^{\dim F}) = F(x)$. Using the exponential map $\exp : TM \to M$ one can construct uniform charts around each point $x \in \Lambda$ of the form $\xi_x := \exp_x \circ C(x) : B(0,R) \to M$ verifying that for $y, y' \in B(0,R)$ one has that $\frac{1}{K} d(\xi_x(y), \xi_x(y') \leq \|y - y'\| \leq K d(\xi_x(y), \xi_x(y'))$. Here $R > 0$ and $K > 0$ are fixed constant independent of x.

One can, by using the same technique as in the proof of Theorem 3.1 lift the dynamics by extending the map $\tilde{f}_x := \xi^{-1}_{f(x)} \circ f \circ \xi_x : B(0, R/K\|Df_x\|) \to B(0,R)$ to a diffeomorphism $\hat{f}_x : \mathbb{R}^d \to \mathbb{R}^d$ which in coordinates $\mathbb{R}^d = \mathbb{R}^{\dim E} \oplus \mathbb{R}^{\dim F}$ can be expressed as:

$$\hat{f}_x(v,w) = (A_x v + \alpha_x(v,w), B_x w + \beta_x(v,w))$$

where $A_x : \mathbb{R}^{\dim E} \to \mathbb{R}^{\dim E}$ and $B_x : \mathbb{R}^{\dim F} \to \mathbb{R}^{\dim F}$ are linear transformation which by the domination[16] condition satisfy $\|A_x\| < \lambda \|B_x^{-1}\|^{-1}$ for some $\lambda \in (0,1)$ and such that the C^1 size of α_x and β_x is smaller than $\varepsilon \ll 1 - \lambda$.

Now, for a given $x \in \Lambda$ we can consider the sequence $\{f_n\}_n$ of diffeomorphisms of $\mathbb{R}^d$ defined as $f_n = \hat{f}_{f^n(x)}$. For this sequence it is possible to consider the

[14]Notice that this has only sense when the bundle $E \subset T_\Lambda M$ is trivializable (for example, when Λ is totally disconnected). Technically, it would be more correct to write that $\mathcal{D}_E : \Lambda \to \mathrm{Emb}^1(E_1, M)$ such that $\mathcal{D}_E(x)$ is an embedding of $E_1(x)$ in M where $E_1(x)$ denotes the disk of radius 1 in $E(x)$.

[15]This is not strictly true if the bundle is not trivializable over Λ. But we shall ignore this technical (and unimportant) issue.

[16]We remark that here we are assuming for simplicity that the dominated splitting comes with an adapted norm. This is no loss of generality, but the same argument can be adapted not to use it.

space of graphs of Lipschitz functions from $\mathbb{R}^{\dim E}$ to $\mathbb{R}^{\dim F}$. As in the proof of Theorem 3.4 one shows that the graph transform induced by the sequence f_n is a contraction for a suitable metric and so there exists a unique sequence of graphs which is invariant under the sequence $\{f_n\}_n$ and it is indeed by C^1-graphs which are tangent to $\mathbb{R}^{\dim E} \times \{0\}^{\dim F}$ at $(0,0)$.

Sending the intersection of the graphs with $B(0,R)$ by ξ_x to M one obtains the desired embedding and notice that the intersection with $B(0,R/K\|Df_x\|)$ is sent to the next graph since it remains in the place where $\hat{f}_x$ coincides with $\tilde{f}_x$. This concludes the proof.

□

Remark 5.3. This result does not provide uniqueness of the plaque families since there is no natural way to lift f to the functions $\hat{f}_x$. This means that for each choice of lift $\{\hat{f}_x\}_{x\in\Lambda}$ of the dynamics one obtains an a priori different plaque family. However, if there are dynamical conditions, for example if $y \in \mathcal{D}_E(x)(\mathbb{D}^{\dim E})$ and $f^n(y) \in \xi_{f^n(x)}(B(0,R))$ for every $n \geq 0$ then the point y will belong to every sufficiently large plaque family.

Notice that one can perform the graph transform argument by starting with a foliation of a neighborhood of x and obtain locally invariant local foliations which are almost tangent to E (or F). This is done in [BuW$_2$, Section 3] where *fake foliations* are constructed. Those fake foliations have some technical applications (notably to the study of stable ergodicity when the center direction is not integrable). One should not be confused by the existence of these local foliations almost tangent to E since it is possible that the bundle E is not locally integrable at any point of the manifold (see [BuW] for examples).

Exercise. By combining Theorem 5.2 and the ideas used for Theorem 3.3 try to show that Theorem 3.7 is valid for C^1-diffeomorphisms of surfaces if the support of the measure admits a dominated splitting.

The previous exercise is a particular case of a more general result which states that much of Pesin's theory works in the C^1-setting if one assumes domination on the support of the invariant measures. See [AbBC] for precise statements and proofs.

5.3. **Uniform hyperbolicity and partial hyperbolicity.** Consider a compact f-invariant set Λ and assume that Df-preserves a continuous splitting of $T_\Lambda M$ into three bundles of the form:

$$T_\Lambda M = E^s \oplus E^c \oplus E^u$$

where E^s is uniformly contracted, E^u is uniformly expanded and the splittings $E^s \oplus (E^c \oplus E^u)$ and $(E^s \oplus E^c) \oplus E^u$ are dominated. We say that:

- Λ is *uniformly hyperbolic* if $E^c = 0$.
- Λ is *partially hyperbolic* if either E^s or E^u is non-zero.
- Λ is *strongly partially hyperbolic* if both E^s and E^u are non-zero.

The study of diffeomorphisms for which their limit set (more precisely their chain-recurrent set) is uniformly hyperbolic is one of the milestones of study of dynamical systems from the pioneering work of Anosov and Smale in the 60's to the present. Its study has interacted with the study of geometry and topology as well as it has been the starting point to many advances in different areas of mathematics. One of the main tools of its study is the following classical result.

Theorem 5.4 (Shadowing Theorem). *Let $f : M \to M$ be a C^1-diffeomorphism and $\Lambda \subset M$ a compact f-invariant hyperbolic subset. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{z_n\}_n \subset \Lambda$ is an δ-pseudo orbit (i.e. a sequence such that $d(z_{n+1}, f(z_n)) \leq \delta$) there exists a point $y \in M$ such that its orbit ε-shadows $\{z_n\}_n$ (i.e. one has $d(f^n(y), z_n) \leq \varepsilon$). Moreover:*

- *one has that $\delta \to 0$ as $\varepsilon \to 0$,*
- *if ε is small enough, the point y whose orbit shadows $\{z_n\}_n$ is unique,*
- *if there exists $m > 0$ such that $z_{n+m} = z_n$ for all $n \in \mathbb{Z}$ one can choose y to be a periodic orbit of period m,*
- *if Λ is* locally maximal *(i.e. if there exists a neighborhood U of Λ such that $\Lambda = \bigcap_n f^n(U)$) then the point y can be chosen to belong to Λ.*

Sketch The proof follows exactly the same lines as the proof of Theorem 4.6 but it is much easier. Indeed, one chooses uniform charts and applies exactly the same argument as in the proof of Theorem 4.1.

□

It has been necessary to understand the global panorama of dynamical systems to consider weaker notions of hyperbolicity. In some cases, non-uniform hyperbolicity has been the right generalization, but in many others, it turns out that dominated splittings or partial hyperbolicity have been more suitable. They verify the following general theorem in the same lines as the results we present in this notes.

Theorem 5.5 (Stable Manifold Theorem). *Let $f : M \to M$ be a C^1-diffeomorphism and let $\Lambda \subset M$ be a compact f-invariant set with a partially hyperbolic splitting of the form $T_\Lambda M = E^s \oplus E^{cu}$ where the bundle E^s is uniformly contracted. Then, there exists a continuous family $\mathcal{W}^s_{loc} : \Lambda \to \mathrm{Emb}^1(\mathbb{D}^{\dim E^s}, M)$ with the following properties:*

- *(**Tangency:**) for every $x \in \Lambda$ one has that $\mathcal{W}^s_{loc}(x)(0) = x$ and the image of $\mathcal{W}^s_{loc}(x)$ is tangent to $E^s(x)$ at x,*
- *(**Invariance:**) for every $x \in \Lambda$ one has that*
$$f(\mathcal{W}^s_{loc}(x)(\mathbb{D}^{\dim E^s})) \subset \mathcal{W}^s_{loc}(f(x))(\mathbb{D}^{\dim E^s})\,,$$
- *(**Convergence:**) if y is in the image of $\mathcal{W}^s_{loc}(x)$ then $d(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to +\infty$,*
- *(**Uniqueness:**) if one considers for each $x \in \Lambda$ the* strong stable set
$$\mathcal{W}^{ss}_x = \bigcup_n f^{-n}(\mathcal{W}^s_{loc}(f^n(x))(\mathbb{D}^{\dim E^s}))$$
it follows that for $x, y \in \Lambda$ the sets $\mathcal{W}^{ss}_x$ and $\mathcal{W}^{ss}_y$ are injectively immersed submanifolds which are either disjoint or coincide.

Proof. It follows almost directly from Theorem 5.2 and Remark 5.3. Indeed, one can consider any plaques family given by Theorem 5.2 and use the fact that $\|Df_x|_{E^s}\| < \lambda < 1$ to see that the diameter of the forward iterates of the plaques converges exponentially fast to zero. This gives invariance and convergence. Uniqueness follows from the fact that independently on the choice of lift, the plaque families will coincide up to their size (see Remark 5.3) so that when considering the set of points that eventually lie in a plaque one has uniqueness.

□

Remark 5.6. It is possible to show that graph transform argument varies continuously with the diffeomorphism in compact sets so that if $f_n \to f$ then the strong stable manifolds (resp. strong unstable manifolds) for f_n converge in compact subsets to those of f.

5.4. **Normal hyperbolicity and persistence.** It is sometimes useful to perform the graph transform method in a more global way. This is the case in the proof of persistence of normally hyperbolic submanifolds or foliations. We refer the reader to [HPS] or [Ber] for detailed proofs.

Consider $f : M \to M$ a C^1-diffeomorphism and $\Lambda \subset M$ a compact f-invariant set. We shall assume that Λ is *laminated* by an f-invariant lamination $\mathcal{L}$. This means that for each $x \in \Lambda$ there exists a C^1-injectively immersed submanifold $\mathcal{L}(x) \subset \Lambda$ with the following properties:

- if $\mathcal{L}(x) \cap \mathcal{L}(y) \neq \emptyset$ then $\mathcal{L}(x) = \mathcal{L}(y)$,
- if $x_n \to x$ then $\mathcal{L}(x_n)$ converges to $\mathcal{L}(x)$ uniformly in the C^1-topology in compact subsets,
- the map $x \mapsto T_x\mathcal{L}(x) \subset T_xM$ defines a continuous distribution.

The f-invariance means that $f(\mathcal{L}(x)) = \mathcal{L}(f(x))$.

We say that the lamination $\mathcal{L}$ is *normally hyperbolic* if f admits a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ where $E^c(x) = T_x\mathcal{L}(x)$ for every $x \in \Lambda$. Moreover, we say it is *normally expanded* (resp. *normally contracted*) if $E^s = \{0\}$ (resp. $E^u = \{0\}$).

Remark 5.7. Notice that if $\mathcal{L}$ is a lamination by points, normal hyperbolicity of $\mathcal{L}$ is equivalent to have that Λ is uniformly hyperbolic.

Whenever there is a normally hyperbolic lamination, one has the following persistence result:

Theorem 5.8 (Stability of normally hyperbolic laminations)**.** *Let $f : M \to M$ be a C^1-diffeomorphism, leaving invariant a normally hyperbolic lamination $\mathcal{L}$ on a compact set Λ. Then, there exists a C^1-neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ there exists a compact g-invariant set Λ_g close to Λ such that:*

- *(**Continuation of leaves:**) for every $x \in \Lambda$ there exists a manifold L_g^x diffeomorphic to $\mathcal{L}(x)$ such that if one considers an immersion $i_x : \mathcal{L}(x) \hookrightarrow M$ there is an immersion $i_x^g : L_g^x \to M$ (possibly no longer injective) such that i_x and i_x^g are C^1-close and L_g^x is everywhere tangent to E_g^c (the continuation of the bundle E^c of f for g in Λ_g),*
- *(**Invariance:**) one has that $f(i_x^g(L_g^x)) = i_{f(x)}^g(L_g^{f(x)})$,*
- *(**Continuity:**) The leafs $i_x^g(L_g^x)$ with $x \in \Lambda$ saturate Λ_g and vary continuously in the C^1-topology in compact subsets.*

The idea of the proof is to perform a graph transform argument in an entire neighborhood of the immersion. This involves unwrapping the immersion to an abstract immersion into a neighborhood of the leaf which depends on the point and then applying arguments very similar to the ones we have already done albeit more technical.

This result is not completely satisfactory since in principle leafs of the new "lamination" could merge. One sometimes calls this *branching laminations* (sometimes, they are useful for some purposes, see [BuI, Pot_5, HP, HP_2] for use of this notion).

Under a technical condition (which is always satisfied in case the lamination can be extended to a neighborhood into a C^1-foliation) it is possible to improve Theorem 5.8 to have a true lamination for diffeomorphisms close to f. This condition is known by the name of *plaque-expansiveness* and we refer the reader to [HPS] and [Ber] for more information about it. We also refer the reader to [BuW] for information on the related notion of *dynamical coherence.*

We make the following remarks on Theorem 5.8 since we shall not enter in the details of its proof. The first remark is that to be able to perform a global graph transform one uses strongly the fact that the dynamics are C^0-close (not only that the invariant bundles are close), this can be noticed by the fact that (a strong version of Theorem 5.8, the one appearing in [HPS]) implies that after C^1-perturbation, the f-invariant foliation remains homeomorphic to the initial one while there might be very different topological type of foliations which are tangent to closeby distributions (just think about linear foliations on tori). The other remark is that even in the simplest case of a closed submanifold $N \subset M$ which is normally hyperbolic, the graph transform must be performed with some care since does not have a priori a fixed point on which to "center" the graph transform argument. We refer the reader to [BerB] for a short proof in this particular and easier case.

5.5. **Reducing the dimension.** Possibly, the most important information given by the existence of a dominated splitting or of the existence of a partially hyperbolic splitting comes with the fact that Theorem 5.2 allows one to "reduce the dimension" of the study. In general, if one has a strong partially hyperbolic splitting, one can use Theorem 5.2 to reduce the situation to a kind of skew-product over a hyperbolic set, at least, one can think the skew-product over a hyperbolic set as a *toy model* for the general situation. This approach has been very successful when $\dim E^c = 1$ (see [Cr]).

However, there are some cases where the reduction of dimension is even more drastic, instead of obtaining a sequence of maps of a lower dimensional manifolds, one can in some cases deal with a unique one. This is the case when the dynamics one is interested in lives in a normally hyperbolic submanifold. As we have seen, this is a robust property, and we shall quickly review in this subsection a result due to Bonatti and Crovisier ([BoC2]) which allows to detect this situation.

Let us state their result.

Theorem 5.9 (Bonatti-Crovisier)**.** *Let Λ be a compact f-invariant set admitting a partially hyperbolic splitting of the form $T_\Lambda M = E^{cs} \oplus E^u$. Assume moreover that for every $x \in \Lambda$ one has that $\mathcal{W}^{uu}(x) \cap \Lambda = \{x\}$. Then, there exists a C^1-submanifold $\Sigma \subset M$ containing Λ and tangent to E^{cs} at every point of Λ such that it is locally invariant (i.e. one has that $f(\Sigma) \cap \Sigma$ is a neighborhood of Λ relative to Σ).*

Exercise. Show that if Λ is partially hyperbolic and it is contained in such a submanifold, then one has that $\mathcal{W}^{uu}(x) \cap \Lambda = \{x\}$ for every $x \in \Lambda$.

5.6. **When can you guarantee the existence of a dominated splitting?** One has the classical cone-field criteria (see [BoGo]) which ensures domination and can be checked with only finitely many iterates (notice that the explicit bundles depend on the complete orbit of the point). This criteria is the one used to prove Proposition 5.1.

There are also criteria to ensure domination when one has information on certain robust properties of the diffeomorphism, a classical result in this line is [BoDP] (see also [BoDV, Chapter 7]). Also, in the lines of a celebrated conjecture due to Jacob Palis, one knows that far from homoclinic tangencies, the dynamics is *partially hyperbolic* (see [CrSY] and references therein).

In the same spirit as the critical points of Benedicks-Carleson, for surface dynamics there exists the critical point criteria to admit dominated splitting first introduced in [PuRH] and further improved by Crovisier and Pujals [CrPu] (see also [Va] for developments in the holomorphic setting).

6. Attractors and the geometry of unstable laminations

In this section we give a glimpse in further topics which use the tools developed in this notes. They represent a very biased choice based on the author's interests.

The main point is to show some of the results of the notes in "action". First, we shall explain how (with help of some results we will just cite) the ideas in the text allow to show that in dimension 2 there is an open and dense subset of diffeomorphisms in the C^1 topology admitting a hyperbolic attractor. This result is part of Araujo's thesis [Ara]. His proof had a gap, and the result became folklore after the results of Pujals-Sambarino ([PuS]). We shall present the proof that appeared in [Pot] (which uses [PuS] but also some other recent results, notably [BoC]).

After we have presented the proof of this result in dimension 2, we shall try to present quickly (with much less details) a recent joint result with Sylvain Crovisier and Martín Sambarino on finiteness of attractors for certain differentiable dynamics which explores the geometry of the strong unstable manifolds.

We refer the reader to [Pot_4, Chapter 3] for a wider panorama on attractors for differentiable dynamics. We also strongly recommend [Cr_3] for a more global point of view of differentiable dynamics on manifolds with plenty of pertinent references. At this point we wish to point out the important influence of the work of Mañé in this type of results, we mention in particular two landmark papers of his [M_1, M_3]. Also, recently we have written some notes with S. Crovisier which complement and extend the material presented here [CrPo].

6.1. **Some preliminaries.**

We start by introducing some preliminaries in the study of differentiable dynamics. We consider C^1-diffeomorphism f of a closed d-dimensional manifold M.

A *topological attractor* is a compact invariant set Λ such that there exists an open set U verifying $f(\overline{U}) \subset U$ and $\Lambda = \bigcap_{n\geq 0} f^n(\overline{U})$.

Exercise. Assume that Λ is a partially hyperbolic topological attractor with splitting $T_\Lambda M = E^{cs} \oplus E^u$ where E^u is uniformly expanded. Show that:

- The set Λ is saturated by strong unstable manifolds $\mathcal{W}^{uu}$ (i.e. the strong stable manifolds for f^{-1}, c.f. Theorem 5.5).
- There exists an ergodic invariant measure μ such that the sum of its Lyapunov exponents is ≤ 0. As a consequence, μ has at least one strictly negative Lyapunov exponent.

In fact, one can see that the first assertion of the second item does not need the fact that Λ is partially hyperbolic.

Topological attractors are not completely satisfying, for example, it always holds that the whole manifold M is a topological attractor. In general, one adds some sort of indecomposability hypothesis to the definition of attractor[17]. We say that Λ is an *attractor* for f if it is a topological attractor and $f|_\Lambda$ is transitive. The *basin* of Λ is the set of points whose omega-limit set is contained in Λ. In the case where Λ is an attractor it is $\bigcup_n f^{-n}(U)$.

Exercise. Show that if $\Lambda \subset M$ is an uniformly hyperbolic attractor of f then:

- its basin is an open set of M,
- there exists a neighborhood $\mathcal{U}$ of f and a neighborhood U of Λ such that for $g \in \mathcal{U}$ one has that $g(\overline{U}) \subset U$ and $\Lambda_g = \bigcap_n g^n(U)$ is a uniformly hyperbolic attractor.

Attractors do not always exists (even for C^r-generic dynamics, see [BoLY, Pot_2]). So, one sometimes uses the notion of *quasi-attractors*. We say that a compact f-invariant set is a *quasi-attractor* if:

- (**Intersection of topological attractors:**) there exist a basis of neighborhoods U_n of Λ such that $f(\overline{U_n}) \subset U_n$,
- (**Indecomposability:**) if U is an open set such that $f(\overline{U}) \subset U$ and $\Lambda \cap U \neq \emptyset$ then $\Lambda \subset U$.

The second hypothesis is equivalent to Λ being chain-transitive which we shall not define here. A remarkable result due to Bonatti and Crovisier states the following:

Theorem 6.1 (Bonatti-Crovisier [BoC]). *There exist a residual (i.e. G_δ-dense) subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{G}$ then:*

- *There exists a residual subset $R_f \subset M$ such that for every $x \in R_f$ the omega-limit set of x for f is contained in a quasi-attractor.*
- *If a quasi-attractor Λ contains a periodic point p then it coincides with its* homoclinic class *$H(p)$ (i.e. the closure of the transverse intersections between the orbits of $W^s(p)$ and $W^u(p)$).*

Exercise. Let p be a hyperbolic saddle and $H(p)$ its homoclinic class. Show that $f|_{H(p)}$ is transitive.

6.2. **Attractors in surfaces.** The point of this subsection is to explain the following result:

Theorem 6.2 (Araujo [Ara]). *For a given closed surface M, there exists a residual subset $\mathcal{G}_A$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{G}_A$ then:*

- *either there are infinitely many attracting periodic points (sinks),*
- *either there are finitely many uniformly hyperbolic attractors whose basins cover an open and dense subset (of full Lebesgue measure) of M.*

By the robustness properties of hyperbolic attractors one deduces as a consequence that there exists an open and dense subset of $\mathrm{Diff}^1(M)$ for which there exists hyperbolic attractors. This contrasts with the situation in higher dimensions ([BoLY]).

---

[17]We warn the reader that there are plenty of possible definitions of attractors, the one we choose, even if quite common, is far from being the unique.

Let us explain the main ideas of the proof. Since having infinitely many sinks is a G_δ-property, we shall assume that f cannot be approximated by a diffeomorphism with infinitely many sinks. Moreover, we can without loss of generality assume that $f \in \mathcal{G}$ of Theorem 6.1 and that the number of sinks that f has is constant[18] in a neighborhood of f.

Now let Λ be a quasi-attractor for f. Notice that since $f \in \mathcal{G}$ such a quasi-attractor exists. We shall assume that Λ is not a sink, otherwise there is nothing to prove. We must show that Λ is uniformly hyperbolic. The key point is to show that Λ admits a dominated splitting since in that case it follows as a consequence of the results of [PuS] that it is uniformly hyperbolic.

So let us show:

Proposition 6.3. Λ *admits a dominated splitting.*

PROOF. This proof is in the lines of what is discussed in subsection 5.6.

Since there exists a neighborhood $\mathcal{U}$ of f such that for $g \in \mathcal{U}$ one has that g has finitely many sinks, one can assume that one cannot create a sink in a neighborhood U of Λ by perturbing f.

We use first that there must exist a measure μ supported in Λ whose sum of Lyapunov exponents is ≤ 0. To show that f admits a dominated splitting in the support of μ we use classical arguments (see for example [AbBC]) which imply that otherwise one can create a sink in an arbitrarily small neighborhood of the support of μ by a small perturbation of f.

Notice that in principle, the support of μ may be smaller than Λ itself. However, we know that f admits a dominated splitting on the support of μ and that the sum of Lyapunov exponents is ≤ 0. This implies that μ is hyperbolic, since otherwise, using Theorem 3.3 one would get a sink[19]. Even if f is only C^1, since it admits a dominated splitting in the support of μ one deduces that one can apply Theorem 4.6 to show that Λ contains periodic points and thus, using again that $f \in \mathcal{G}$ conclude that Λ is the homoclinic class of a periodic point such that the sum of its Lyapunov exponents is ≤ 0.

A classical argument of *transitions* (see [BoDV, Chapter 7]) implies that there is a dense set of periodic points in Λ such that the sum of Lyapunov exponents is ≤ 0. If there were not a dominated splitting in Λ, then a small perturbation (see again [BoDV, Chapter 7]) allows one to construct a sink. This concludes.

□

As we mentioned, by [PuS] this implies that Λ is a hyperbolic attractor. It remains to show that there are finitely many. But the argument is very similar, if this were not the case, one would obtain a sequence $\Lambda_n \to \Gamma$ of hyperbolic attractors. One can take the measures μ_n with Lyapunov exponents adding ≤ 0 and obtain a similar measure μ in Γ. This allows to show that Γ admits a dominated splitting (one needs to use a stronger version of [BoC] which deals with classes which are not necessarily quasi-attractors) and therefore is uniformly hyperbolic according to [PuS]. This is a contradiction and one obtains finiteness.

[18]The number of sinks of a diffeomorphism is a semicontinuous function to the natural numbers and ∞. Therefore, it is continuous (and therefore locally constant) in a residual subset.

[19]There could be a zero Lyapunov exponent, but again using Mañé's ergodic closing lemma and Franks Lemma ([AbBC]) one would create a sink by small perturbation.

The fact that the basin is open and dense is direct from the fact that the basins of hyperbolic attractors is open and Theorem 6.1. To show that the basins cover a full Lebesgue measure subset one has to use a semicontinuity argument on the size of basins and use the fact that for C^2-diffeomorphisms hyperbolic sets have zero Lebesgue measure. See [Ara] or [San] for details.

6.3. **Partially hyperbolic attractors with one dimensional center.** We explain here part of a work in progress joint with S. Crovisier and M. Sambarino which studies the geometry of partially hyperbolic sets saturated by strong unstable manifolds. This study is motivated by the fact that attractors are saturated by strong unstable manifolds.

Together with recent results of [CrPuS] and [CrSY] (which use completely different techniques) our main result gives as a consequence the following result which is a step towards the understanding of dynamics far from homoclinic tangencies. It also improves (in dimension 3) a result announced[20] in [BoGLY] (though their result holds in any dimension).

Theorem 6.4 ([CrPoS], [CrPuS], [CrSY]). *Let M be a 3-dimensional manifold. Then, there exists an open and dense subset $\mathcal{U}$ of $\text{Diff}^1(M)$ such that if $f \in \mathcal{U}$ then:*

- *either f has robustly finitely many quasi-attractors,*
- *or f can be C^1-approximated by a diffeomorphism g which has a hyperbolic periodic point p whose stable and unstable manifolds intersect non-transversally (i.e. g has a* homoclinic tangency*).*

Let us call $\text{HT}^1(M)$ the set of diffeomorphisms of M admitting a homoclinic tangency. Putting together the results of [CrPuS] and [CrSY] one can show[21] the following:

Theorem 6.5 (Crovisier,Pujals,Sambarino,D.Yang). *There exists a residual subset $\mathcal{G}_{CPSY}$ of $\text{Diff}^1(M) \setminus \overline{\text{HT}^1(M)}$ such that if $f \in \mathcal{G}_{CPSY}$ one has the following property:*

- *there exists a* filtration *$\emptyset = U_0 \subset U_1 \subset \ldots \subset U_{k-1} \subset U_k = M$ of open subsets such that $f(\overline{U_i}) \subset U_i$ and such that for every i, if $\Lambda_i = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U_i} \setminus U_{i-1})$ then Λ_i verifies one of the following three possibilities:*
  - *Λ_i is a sink,*
  - *Λ_i is a source, or,*
  - *Λ_i admits a strongly partially hyperbolic splitting $T_{\Lambda_i}M = E^s \oplus E^c \oplus E^u$ where both E^s is uniformly contracted and non-zero, E^u is uniformly expanded and non-zero and E^c admits a subdominated splitting into one-dimensional bundles.*

The advantage of working in dimension 3 is that we always know that the dimension of E^c is at most 1. The strategy of the proof is showing that each Λ_i can contain at most finitely many quasi-attractors, so, we are reduced to showing that in a compact f-invariant subset with a strong partially hyperbolic splitting with

[20] In [BoGLY] they show that there exists a residual subset $\mathcal{G}$ of diffeomorphisms far away from homoclinic tangencies such that if $f \in \mathcal{G}$ then all quasi-attractors of f are isolated from each other (but might in principle accumulate in a set which is not a quasi-attractor). They call *essential attractors* to such quasi-attractors since it can be shown that their basin contains a residual subset of a neighborhood.

[21] These result also rely on other results, see [Cr_3] for a more complete account.

$\dim E^c = 1$ one can have at most finitely many quasi-attractors. This is also a consequence of the results of [CrPoS] and what we shall try to briefly explain in what follows.

6.3.1. *Minimal $\mathcal{W}^{uu}$-saturated sets.* Let $f : M \to M$ be a C^1-diffeomorphism. Consider a set Λ which is of the form $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U} \setminus V)$ where U and V are open subsets of M verifying that $f(\overline{U}) \subset U$ and $f(\overline{V}) \subset V$.

The open set $U \setminus \overline{V}$ is a neighborhood of Λ and we know that if a point $x \in U \setminus \overline{V}$ verifies that $f(x) \notin U \setminus \overline{V}$ (resp. $f^{-1}(x)$) then $f^n(x) \notin U \setminus \overline{V}$ for all $n \geq 1$ (resp. $n \leq -1$). This allows one to prove the following.

Exercise. Show that if a quasi-attractor Q intersects $U \setminus \overline{V}$ then $Q \subset \Lambda$.

We shall assume moreover that Λ admits a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ where both E^s and E^u are non-zero and such that $\dim E^c = 1$. Our goal is to show that there are finitely many quasi-attractors in Λ. Recall that quasi-attractors are $\mathcal{W}^{uu}$-saturated.

Exercise. Use Theorem 5.5 to show that if $\{Q_n\}$ is a sequence of quasi-attractors in Λ converging to Θ in the Hausdorff topology, then $\Theta \subset \Lambda$ is $\mathcal{W}^{uu}$-saturated.

We say that a (non-empty) compact f-invariant and $\mathcal{W}^{uu}$-saturated subset Γ of Λ is a *minimal $\mathcal{W}^{uu}$-saturated set* if for every Γ' strictly contained in Γ which is compact f-invariant and $\mathcal{W}^{uu}$-saturated one has that $\Gamma' = \emptyset$. That is, a subset $\Gamma \subset \Lambda$ is a minimal $\mathcal{W}^{uu}$-saturated set if it is minimal for being compact, f-invariant and $\mathcal{W}^{uu}$-saturated.

Exercise. Show that if Λ contains a non-empty compact $\mathcal{W}^{uu}$-saturated subset Λ', then there exist minimal $\mathcal{W}^{uu}$-saturated sets. Moreover, show that every quasi-attractor $Q \subset \Lambda$ contains at least one minimal $\mathcal{W}^{uu}$-saturated set.

Notice that if Q and Q' are two different quasi-attractors then $Q \cap Q' = \emptyset$. Therefore, there are fewer quasi-attractors in Λ than there are minimal $\mathcal{W}^{uu}$-saturated sets. The main result on [CrPoS] is the following.

Theorem 6.6 ([CrPoS])**.** *There is an open and dense subset $\mathcal{O}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{O}$ and Λ a compact f-invariant set admitting a strong partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ with $\dim E^c = 1$ then Λ contains at most finitely many $\mathcal{W}^{uu}$-saturated sets.*

The proof of this Theorem has two stages. First, a perturbation result which provides a geometric property of $\mathcal{W}^{uu}$-saturated laminations for diffeomorphisms in a C^1-open and dense subset of diffeomorphisms. The second stage is to show that this geometric property forbids the minimal $\mathcal{W}^{uu}$-saturated sets to accumulate and since (unlike quasi-attractors) minimal $\mathcal{W}^{uu}$-saturated sets are closed in the Hausdorff topology, this concludes.

6.3.2. *Geometry of strong connections.* We start by explaining the consequence of our perturbation result. The proof of this result is the most delicate part of [CrPoS] and it is the part which we shall omit in this notes. It is not (only) because of laziness but because the techniques are farther away from the interests of this notes.

The statement is the following:

Theorem 6.7. *There exists a G_δ-dense subset $\mathcal{G}$ of* $\text{Diff}^1(M)$ *such that for every $f \in \mathcal{G}$ and $\Lambda' \subset M$ a compact f-invariant partially hyperbolic set which is $\mathcal{W}^{uu}$-saturated and for every $r, r't, \gamma > 0$ sufficiently small, there exists $\delta > 0$ with the following property.*

If $x, y \in \Lambda'$ satisfy $y \in \mathcal{W}^{ss}(x)$ and $d_s(x,y) \in (r, r')$, then there is $x' \in \mathcal{W}^{uu}_t(x)$ such that:

$$d(\mathcal{W}^{ss}_\gamma(x'), \mathcal{W}^{uu}_\gamma(y)) > \delta$$

By d_s we refer to the distance inside $\mathcal{W}^{ss}$ and $\mathcal{W}^\sigma_\varepsilon(x)$ ($\sigma = ss, uu$) to denote the ε-ball around x in $\mathcal{W}^\sigma(x)$ with the intrinsic metric. We could have used $\mathcal{W}^\sigma_{loc}(x)$ in each of the places, but the way we have formulated is a bit more explicit.

Using the continuity of the strong manifolds with respect to the diffeomorphism (see Remark 5.6), one sees that at a given scale (i.e. if one fixes the values of r, r', t and γ), this property holds for small perturbations of $f \in \mathcal{G}$ and therefore, in an open and dense subset of $\text{Diff}^1(M)$.

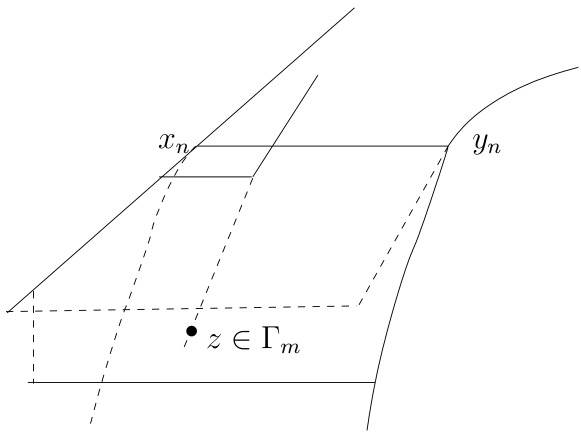

FIGURE 4. The stable manifolds of the minimal sets must intersect.

6.3.3. *Finiteness of minimal saturated sets.* Now, we use Theorem 6.7 as well as the results in the previous sections of this notes to conclude the proof of Theorem 6.6.

Let $\Lambda' \subset \Lambda$ be a compact f-invariant and $\mathcal{W}^{uu}$-saturated set. We must show that there are finitely many minimal $\mathcal{W}^{uu}$-saturated sets in Λ'.

Exercise. Show that if Λ has infinitely many minimal $\mathcal{W}^{uu}$-saturated sets, then there exists $\Lambda' \subset \Lambda$ compact, f-invariant and $\mathcal{W}^{uu}$-saturated containing infinitely many minimal $\mathcal{W}^{uu}$-saturated sets.

The following remark will be important in the proof.

Exercise. Show that if $\Gamma, \Gamma' \subset \Lambda'$ are different minimal $\mathcal{W}^{uu}$-saturated sets, then their stable manifolds are disjoint. That is, if Γ and Γ' are minimal $\mathcal{W}^{uu}$-saturated sets and there exists $x \in \Gamma$ such that there exists $y \in \Gamma'$ such that $d(f^n(x), f^n(y)) \to 0$ as $n \to +\infty$ then $\Gamma = \Gamma'$.

First, we shall show the following.

Proposition 6.8. *In Λ' there are at most finitely many minimal $\mathcal{W}^{uu}$-saturated sets Γ with the property that for every $x \in \Gamma$ one has that $\mathcal{W}^{ss}(x) \cap \Gamma = \{x\}$.*

Proof. Assume by contradiction that there are infinitely many such subsets and denote them as $\{\Gamma_n\}_n$. Notice that thanks to Theorem 5.9 we know that for each n one has that Γ_n is contained in a locally invariant submanifold Σ_n tangent to $E^c \oplus E^u$ at each point of Γ_n.

Notice moreover that there exists $h > 0$ such that $h_{top}(f|_{\Gamma_n}) > h$ for every n. This follows from the following argument: consider a finite covering of Γ_n by balls of radius ε where ε is small enough (independent on n) so that any disk tangent to a small cone around E^u of diameter 1 contains at least two disks of radius ε contained in different balls of the covering. Now, we know that given any disk tangent to a small cone around E^u its iterates grow so that the internal radius multiplies by an uniform amount (independent of n). We can choose such a disk D to be contained in Γ_n (since it is $\mathcal{W}^{uu}$-saturated). We get that for some k_0 (independent of n), the image $f^{k_0}(D)$ contains two such disks. Therefore, inside D one has that in k_0 iterates we duplicate the number of "different" orbits and therefore the entropy of f in Γ_n is larger than $\frac{1}{k_0}\log 2$ (independent of n).

Now, using the variational principle and Ruelle's inequality (Theorem 4.3) for f^{-1} we obtain that Γ_n has a measure μ_n whose Lyapunov exponent for f^{-1} along E^c (recall that Γ_n "lives" in Σ_n) is larger than h. This means that Γ_n has points whose stable manifold has uniform size[22] along $E^s \oplus E^c$. This implies that for every n, there is an open ball B_n of uniform volume such that no other Γ_m can intersect for $m \neq n$. This is impossible if there are infinitely many Γ_n.

□

Now we are in conditions to complete the proof of Theorem 6.6. Consider $f \in \mathcal{G}$ given by Theorem 6.7 (or in a small neighborhood so that the same properties hold at a given scale).

Assume by contradiction that there are infinitely many different minimal $\mathcal{W}^{uu}$-saturated sets $\{\Gamma_n\}_n$ in Λ'. By Proposition 6.8 we can assume that for every n there exists $x_n \in \Gamma_n$ such that $\mathcal{W}^{ss}(x_n) \cap \Gamma_n \neq \{x_n\}$.

By iteration, we can assume that we have points $x_n, y_n \in \Gamma_n$ such that $y_n \in \mathcal{W}^{ss}(x_n)$ and $d_s(x_n, y_n) \in (r, r')$ for some $r' > \Delta r$ where $\Delta \geq \max_x\{\|D_x f^{\pm 1}\|\}$.

Then, these pairs of points converge to points $x, y \in \Lambda'$ which belong to the same local stable manifold ($y \in \mathcal{W}^{ss}(x)$ and $d_s(x,y) \in (r,r')$). Since the strong unstable manifolds get separated by projection by stable holonomy, it is possible to show[23] that this configuration forces the strong stable manifold of one of the Γ_n to intersect some other Γ_m (see Figure 4) contradicting the fact that the minimal sets where different. This concludes.

References

[AbBC] F. Abdenur, C. Bonatti, S.Crovisier, Non-uniform hyperbolicity for C^1-generic diffeomorphisms, *Israel J. of Math.* **183** (2011), 1-60.

[Ara] A. Araujo, Existencia de atratores hiperbolicos para difeomorfismos de superficies, Ph.D. Thesis, IMPA, (1987).

[22] We have not proved this explicitely, but it follows from the arguments we have done along the notes.

[23] Details to appear soon in [CrPoS].

[Arn] M. C. Arnaud, Hyperbolicity for conservative twist maps of the 2 - dimensional annulus , *Publicaciones Matemáticas del Uruguay* **16** (2016), 1-39.

[AvB] A. Avila, J. Bochi, Trieste lecture notes on Lyapunov exponents, (2008) available at http://www.mat.uc.cl/ jairo.bochi/docs/trieste.pdf

[AvB_2] A. Avila, J. Bochi, On the subaditive ergodic theorem, *unpublished* (2009) available at http://www.mat.uc.cl/ jairo.bochi/docs/kingbirk.pdf

[AvCW] A. Avila, S. Crovisier, A. Wilkinson, Diffeomorphisms with positive metric entropy,*Preprint* arXiv:1408.4252.

[BaP] L. Barreira, Y. Pesin, *Nonuniform hyperbolicity*, Cambridge University Press (2007).

[BeC] M. Benedicks and L. Carleson. The dynamics of the Hénon map. *Ann. Math.* **133** (1991) 73Â–169.

[Ber] P. Berger, Persistence of laminations, *Bull Braz Math Soc, New series* **41** 2, (2010) p. 259-319.

[Ber_2] P. Berger, Abundance of non-uniformly hyperbolic Hénon like endomorphisms, *Preprint* arXiv:0903.1473.

[BerB] P. Berger, A. Bounemoura, A geometrical proof of the persistence of normally hyperbolic submanifolds, *Dynamical systems, An International Journal* **28** 4 (2013).

[BeKZ] P. Bernard, V. Kaloshin, K. Zhang, Arnold diffusion in arbitrary degrees of freedom and crumpled 3-dimensional normally hyperbolic invariant cylinders, *Preprint* arXiv:1112.2773

[Bo] J. Bochi, The multiplicative ergodic theorem, *unpublished* (2008) available at http://www.mat.uc.cl/ jairo.bochi/docs/oseledets.pdf

[BoGo] J. Bochi and N. Gourmelon, Some characterizations of domination, *Mathematische Zeitschrift* **263** (2009), no. 1, pp. 221-231.

[BoC] C. Bonatti, S. Crovisier, Récurrence et généricité, *Inventiones Math.* **158** (2004), 33Â–104.

[BoC2] C. Bonatti, S. Crovisier, Center manifolds for partially hyperbolic set without strong unstable connections, to appear in *Journal of the IMJ.*

[BoCS] C. Bonatti, S. Crovisier, K. Shinohara, The $C^{1+\alpha}$ hypothesis in Pesin theory revisited, *Journal of Modern Dynamics* **7** (2013), 605-618.

[BoDP] C. Bonatti, L. Díaz and E. Pujals, A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources, *Annals of Mathematics* **158**(2), (2003) 355-418.

[BoDV] C. Bonatti, L. Díaz and M. Viana, *Dynamics Beyond Uniform Hyperbolicity. A global geometric and probabilistic perspective*, Encyclopaedia of Mathematical Sciences **102**. Mathematical Physics III. Springer-Verlag (2005).

[BoGLY] C. Bonatti, S. Gan, M. Li and D. Yang, *In preparation.*

[BoLY] C. Bonatti, M. Li, D. Yang, On the existence of attractors, *Transactions of the AMS* **365** 3 (2013) 1369Â–1391.

[Br] M. Brin, On dynamical coherence, *Ergodic Theory and Dynamical Systems* **23** (2003) 395–401.

[BrP] M. Brin, Y. Pesin, Partially hyperbolic dynamical systems, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 170Â–212.

[BuI] D. Burago and S. Ivanov, Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups. *Journal of Modern Dynamics* **2** (2008) 541Â–580.

[BuW] K.Burns and A. Wilkinson, Dynamical coherence and center bunching, *Discrete and Continuous Dynamical Systems A* (Pesin birthday issue) **22** (2008) 89-100.

[BuW_2] K. Burns and A. Wilkinson, On the ergodicity of partially hyperbolic systems, *Ann. of Math.* **171** (2010) 451-489.

[Cr] S. Crovisier, Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems, *Annals of Math.* **172** (2010), 1641-1677.

[Cr_2] S. Crovisier, Dynamics of surface diffeomorphisms, *Course Draft Notes* (2013).

[Cr_3] S. Crovisier, Dynamics of C^1-diffeomorphisms: global description and prospects for classification, to appear in Proceedings of the ICM (2014) arXiv:1405.0305.

[CrPo] S. Crovisier, R. Potrie, Introduction to partially hyperbolic dynamics, *Lecture Notes* for a minicourse at ICTP School and Workshop on Dynamical Systems (2015) available in the web page of the conference.

[CrPoS] S. Crovisier, R. Potrie and M. Sambarino, Geometric properties of partially hyperbolic attractors, *In preparation.*

[CrPu] S. Crovisier, E. Pujals, *In preparation.*

[CrPuS] S. Crovisier, E. Pujals, M. Sambarino, Hyperbolicity of extreme bundles, *In preparation.*
[CrSY] S. Crovisier, M. Sambarino, D. Yang, Partial hyperbolicity and homoclinic tangencies, to appear in *Journal of the EMS* arXiv:1103.0869.
[DeM] G. Del Magno, R. Markarian, On the Bernoulli property of planar hyperbolic billiards, *Preprint* http://premat.fing.edu.uy/ 2014/177.
[Ge] K. Gelfert, Horseshoes for diffeomorphisms preserving hyperbolic measures, to appear in *Math. Z.*
[Gou] N. Gourmelon, Adapted metrics for dominated splittings, *Ergodic Theory and Dynamical Systems* **27** (2007), pp 1839-1849.
[GK] M. Guardia, V. Kaloshin, Orbits of nearly integrable systems accumulating to KAM tori, *Preprint* arXiv:1412.7088.
[HP] A. Hammerlindl and R. Potrie, Pointwise partial hyperbolicity in 3-dimensional nilmanifolds, *Journal of the London Math. Society* **89** (3) (2014): 853-875.
[HP_2] A. Hammerlindl and R. Potrie, Partial hyperbolicity in 3-dimensional manifolds with solvable fundamental group, *Journal of Topology* **8** (3), (2015) 842-870 .
[HPS] M. Hirsch, C. Pugh and M. Shub, Invariant Manifolds, *Springer Lecture Notes in Math.*, **583** (1977).
[KZ] V. Kaloshin, K.Zhang, Normally hyperbolic invariant manifolds near strong double resonance, *Preprint* arXiv:1202.1032
[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press (1995).
[LeR] F. Le Roux, L'ensemble de rotation autour d'un point fixe d'homÃ©omorphisme de surface, *Astérisque* **350** (2013).
[M_1] R. Mañé, Contributions to the stability conjecture, *Topology* **17** (1978), 383–396.
[M_2] R. Mañé, A proof of Pesin's entropy formula, *Ergodic Theory and Dynamical Systems* **1** (1981) 95-102.
[M_3] R. Mañé, An ergodic closing lemma, *Annals of Math.* **116** (1982), 503–540.
[M_4] R. Mañé, *Ergodic theory and differentiable dynamics*, Springer-Verlag (1983).
[Pot] R. Potrie, A proof of the existence of attractors in dimension 2, *Preprint* (2009). Not intended for publication, available in my web page.
[Pot_2] R. Potrie, Wild Milnor attractors accumulated by lower dimensional dynamics, *Ergodic Theory and Dynamical Systems* **31** 1 (2014) 236-262.
[Pot_3] R. Potrie, On the work of Jorge Lewowicz on expansive systems, *Publicaciones Matematicas del Uruguay* **14** (2014) 6-22.
[Pot_4] R. Potrie, Partial hyperbolicity and attracting regions in 3-dimensional manifolds, Thesis (2012) arXiv:1207.1822.
[Pot_5] R. Potrie, Partial hyperbolicity and foliations in $\mathbb{T}^3$, *Journal of Modern Dynamics* **9** (2015) 81–121.
[PuRH] E. Pujals, F. Rodriguez Hertz, Critical points for surface diffeomorphisms, *Journal of Modern Dynamics* **1** (4) (2007) 615–648.
[PuS] E. Pujals, M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Annals of Math.* **151** (2000), 961Â–1023.
[San] B. Santiago, Hyperbolicidade essencial em superficies, Dissertacao de Mestrado UFRJ (2011).
[Sar] O. Sarig, Symbolic dynamics for surface diffeomorphisms with positive entropy, *Journal of the AMS* **26** (2013) 341–426.
[Sh] M. Shub, *Global stability of dynamical systems*, Springer-Verlag (1987).
[Va] F. Valenzuela, On critical points for two dimensional holomorphic systems, to appear in *Ergodic Theory and Dynamical Systems.*

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