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HOLOMORPHIC CURVES AND CELESTIAL MECHANICS

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Abstract. This expository article has two purposes. The first is to explain the connection between Poincaré's work on Celestial Mechanics and the Arnold Conjecture on the minimal number of fixed points of Hamiltonian diffeomorphisms. The second is to outline how holomorphic curve methods can be used to study Hamiltonian dynamics.

1. From Celestial Mechanics to Arnold Conjecture

In order to reach the origins of the problems that motivated Floer Theory we need to revisit the end of the XIX century. During this time Celestial Mechanics served as stage for Poincaré to implement some of the revolutionary ideas that laid the foundations of the field of Dynamical Systems. Poincaré was led to make a fundamental statement, known today as the Poincaré-Birkhoff Theorem [32], proved by Birkhoff [5] in 1913. It is a fixed point theorem for certain area-preserving annulus homeomorphisms, the precise statement will be recalled below. For decades mathematicians looked for a better understanding of the theorem, for generalizations to higher dimensions and to other phase spaces. We had to wait until the 1960's when ideas of Arnold finally allowed for proper understanding and generalization. The main purpose of this introduction is to provide more details about this remarkable chain of events.

Poincaré investigated a simplification of the 3-body problem: the *planar circular restricted* 3*-body problem* (PCR3BP). Two massive particles evolve in a relatively circular trajectory of the 2-body problem. A third particle (satellite) moves in the same plane of the first two, under their influence according to Newton's Law of Gravitation. However, the satellite is assumed not to disturb the movement of the massive particles. It turns out that in a rotating coordinate system, where the massive particles remain fixed, the satellite's movement is described by an *autonomous* Hamiltonian; see Section 4 for precise formulas.

Low energy levels (below lowest critical level) have three connected components, they project onto three so-called *Hill regions* of the configuration plane: two of which are bounded neighborhoods of the massive particles while the third is a neighborhood of infinity. We choose one bounded Hill region. The *mass ratio* $\mu \in (0,1)$ is defined as the ratio between the mass of the particle outside the chosen Hill region and the total mass. The Hamiltonian turns out to have a smooth limit as $\mu \to 0$. The limiting Hamiltonian models the *rotating Kepler problem*, which is just the Kepler problem viewed in a rotating coordinate system.

The rotating Kepler problem is integrable, the integrals being the energy and the angular momentum. The Hill region around the light particle degenerates as $\mu \to 0$, while the boundaries of the other Hill regions converge to circles around the limiting position of the heavy particle, which we call *center*. Using integrability it is not difficult to find circular orbits of the rotating Kepler problem. There are three of them: two inside the bounded Hill region, and a third one inside the unbounded Hill region. One of the two circular orbits in the bounded Hill region is *retrograde* while the other is *direct*, according to which they rotate around the center in the opposite or in the same sense of the relative position of the massive particles.

Poincaré was able to find continuations of the circular orbits when $\mu > 0$ is small enough. For this he devised a method, called the *continuation method*, which we now briefly describe. Choose a transverse section at a point of one of the orbits (recall that energy levels are three-dimensional). There is a well-defined local return map, and the orbit corresponds to a fixed point. The transverse section remains, of course, transverse as we slightly perturb μ. If the derivative of the return map at this fixed point is non-singular for $\mu = 0$ then the fixed point is isolated, and the implicit function theorem gives isolated fixed points of the local return map for all $\mu > 0$ small enough. These correspond to continued periodic orbits. They retain the geometric properties of the unperturbed orbits, namely one is retrograde and the other is direct.

Now, using a method introduced by Levi-Civita [31] which we shall illustrate in Section 4, collisions with the massive particle can be regularized to obtain an energy level diffeomorphic to S^3. The movement evolves smoothly there. In doing so, an ambiguity in the representation of states is introduced: the energy level is antipodal symmetric, each state being represented twice as a pair of antipodal points. We ignore this problem for now, keep the ambiguity and say that the regularized dynamics is a two-to-one lift of the unregularized dynamics. The same regularization process applies of course to the rotating Kepler problem. The direct and retrograde orbits lift under this process to a pair of periodic orbits forming a Hopf link. When $\mu = 0$ Poincaré uses the angular momentum to show that these two lifted orbits bound a very special embedded annulus, a so-called *global surface of section* (GSS).

Definition 1.1. Let ϕ^t be a smooth flow on a smooth closed 3-manifold M. A global surface of section for ϕ^t is an embedded compact surface $S \hookrightarrow M$ such that

(i) The boundary ∂S consists of periodic orbits, and the interior $S \setminus \partial S$ is transverse to ϕ^t.
(ii) For every $p \in M \setminus \partial S$ there exist $t_- < 0 < t_+$ such that $\phi^{t_\pm}(p) \in S$.

In the presence of a global surface of section the dynamical properties of the flow ϕ^t get encoded in the first return map

$$\psi : S \setminus \partial S \to S \setminus \partial S \qquad \psi(p) = \phi^{\tau(p)}(p) \tag{1}$$

where the first return time $\tau : S \setminus \partial S \to (0, +\infty)$ is defined as

$$\tau(p) = \inf\{t > 0 \mid \phi^t(p) \in S\}. \tag{2}$$

Poincaré showed that the annular global surface of section for $\mu = 0$ could be continued for $\mu > 0$ small enough. In fact, the continuation method applies to continue its boundary, as explained above. Using a C^∞-small isotopy, one continues the annulus “by hand”. The task is now to show that the continued annulus is again a global surface of section. By transversality, the return map is well-defined on compact subsets of the interior, and globally encodes the dynamics on arbitrarily large compact subsets of the complement of the boundary orbits. The situation near the boundary is more delicate, but it can be nicely controlled by the robust positivity of the transverse rotation numbers of the boundary orbits.

A more detailed discussion about global surfaces of section is found in Section 3, where the connection to holomorphic curves will be made.

Why is it nice to have a global surface of section? Because it allows for two-dimensional methods to come into play and shed light onto the three-dimensional flow. This is precisely what Poincaré did, and in doing so he was able to leave the seed for Symplectic Topology and Floer Theory. Let us explain this point, which is main goal of this introduction.

Using the linearized dynamics of the direct and retrograde orbits, one can smoothly extend the first return map up to boundary. Each boundary component is mapped onto itself. The symplectic nature of the flow allows us to find a smooth 2-form on the annulus which is preserved by the extended return map. This 2-form defines an area form in the interior, but vanishes on the boundary. It turns out that there is a homeomorphism between Poincaré's annulus and $\mathbb{R}/\mathbb{Z} \times [0,1]$ that is smooth in the interior and pulls the 2-form back to the standard area element. Such a conjugating homeomorphism is not differentiable on the boundary.

Thus, Poincaré was led to the problem of studying qualitative properties of area- and orientation-preserving homeomorphisms of the closed annulus that preserve boundary components. Consider such a homeomorphism f on $\mathbb{R}/\mathbb{Z} \times [0,1]$. It can be lifted to a homeomorphism F of $\mathbb{R} \times [0,1]$. If we denote its components by $F(x,y) = (X(x,y), Y(x,y))$ then

$$f(x+\mathbb{Z}, y) = (X(x,y)+\mathbb{Z}, Y(x,y)) \qquad \forall (x,y) \in \mathbb{R} \times [0,1]. \tag{3}$$

Of course, there are infinitely many choices of F, and they all differ by translation by an integer. It follows that

$$X(x+1,y) = X(x,y)+1 \qquad Y(x+1,y) = Y(x,y) \tag{4}$$

holds for every $(x,y) \in \mathbb{R} \times [0,1]$.

Poincaré's seminal contribution to Symplectic Dynamics and Topology was the formulation of the following statement.

Theorem 1.2 (Poincaré's last geometric theorem [32]). *Suppose that*

$$x \mapsto X(x,1) - x \qquad \text{and} \qquad x \mapsto X(x,0) - x$$

are non-vanishing functions with opposite signs. Then F has at least two fixed points which are not integer translations of each other. In particular, they project onto distinct fixed points of f.

This is a truly remarkable statement. Note that $X(x,1) - x$ and $X(x,0) - x$ are 1-periodic functions of x in view of (4). The assumption that they have definite and opposite signs means that f moves boundary components in opposite directions, i.e. f "twists the annulus".

Theorem 1.2 can be strengthened as follows. Let

$$\pi_{\mathbb{R}} : \mathbb{R} \times [0,1] \to \mathbb{R} \qquad (x,y) \mapsto x$$

denote the projection onto the first component, and consider rotation numbers

$$\rho_0 = \lim_{n\to+\infty} \frac{\pi_{\mathbb{R}} \circ F^n(x,0)}{n} \qquad \rho_1 = \lim_{n\to+\infty} \frac{\pi_{\mathbb{R}} \circ F^n(x,1)}{n} \tag{5}$$

of F restricted to boundary components. The *twist condition* asks that

$$\rho_0 \neq \rho_1. \tag{6}$$

A different choice of F will change ρ_0, ρ_1 by addition of a common integer, hence (6) is really a condition on f. The assumptions of Theorem 1.2 force ρ_0 and ρ_1 to have opposite signs, in particular (6) holds.

Theorem 1.3 (Poincaré-Birkhoff)**.** *If $\rho_0 \neq \rho_1$ then for every p/q between ρ_0 and ρ_1 there exist $P, P' \in \mathbb{R} \times [0,1]$ satisfying*

$$F^q(P) = P + (p,0) \qquad F^q(P') = P' + (p,0)$$

Moreover, the orbits of these points project to distinct periodic orbits of the map f. In particular, f admits infinitely many periodic points.

The above form of the theorem forces existence of infinitely many periodic orbits in the PCR3BP when $\mu > 0$ is small enough and the energy is low. This amounts for one of the fist major victories of Poincaré's ideas.

Birkhoff's proof [5] did not shed light onto generalizations to higher dimensions or to different phase spaces. However, Poincaré was already able to prove Theorem 1.2 in special cases, and his arguments did lend to such generalizations. This was remarked by V. I. Arnold in the 1960's; the following discussion is based on [3, appendix 9]. Arnold realized that, in the smooth case, Theorem 1.2 would be consequence of a certain fixed point theorem for a special class of diffeomorphisms of the torus. Arnold's statement can be generalized to all symplectic manifolds. This was, in fact, only one of many statements that came to be known as *Arnold Conjectures*. These conjectures led Floer [10, 11, 12] to develop what is nowadays known as *Floer theory*.

Arnold considered certain diffeomorphisms of the $2n$-dimensional torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. The space $\mathbb{R}^{2n}$ carries a standard symplectic form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

where coordinates are denoted by $(x_1, \dots, x_n, y_1, \dots, y_n)$. Since ω_0 is invariant by translations, it descends to a symplectic form on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ again denoted by ω_0. Arnold considers diffeomorphisms $\Psi : \mathbb{R}^{2n}/\mathbb{Z}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the following properties:

(H1) Ψ is isotopic to the identity.
(H2) Ψ is symplectic: $\Psi^*\omega_0 = \omega_0$.
(H3) Ψ is exact: it admits a lift to $\mathbb{R}^{2n}$ whose center of gravity vanishes.

Let us explain these in more detail. By (H1) the map Ψ lifts via the projection $\mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ to a symplectic diffeomorphism $\widetilde{\Psi}$ of $(\mathbb{R}^{2n}, \omega_0)$ satisfying

$$\widetilde{\Psi}(z+k) = \widetilde{\Psi}(z) + k \qquad \forall z \in \mathbb{R}^{2n},\ \forall k \in \mathbb{Z}^{2n}. \tag{7}$$

Hence we can write $\widetilde{\Psi}(z) = z + \Delta(z)$ for some $\mathbb{Z}^{2n}$-periodic function Δ. The center of mass of $\widetilde{\Psi}$ is defined by the integral

$$\int_{[0,1]^{2n}} \Delta(z) \tag{8}$$

There are of course infinitely many lifts and they all differ by translation by a vector in $\mathbb{Z}^{2n}$. In particular, the same holds for their centers of mass. Hence condition

(H3) is equivalent to saying that the center of mass of any lift of Ψ belongs to $\mathbb{Z}^{2n}$. We may simply refer to Ψ as an *exact symplectic diffeomorphism* of $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega_0)$.

Arnold Conjecture (Particular case): Exact symplectic diffeomorphisms of the standard symplectic torus $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega_0)$ *must have at least* $2n+1$ *fixed points. Morevoer, if all fixed points are non-degenerate then there must be at least* 2^{2n} *fixed points.*

To illustrate the ideas we prove the conjecture in case the exact symplectic diffeomorphism is C^1-close to the identity. Let $\widetilde{\Psi}$ be the unique lift to $\mathbb{R}^{2n}$ that is C^1-close to the identity. Identifying $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$, we can write in components

$$\widetilde{\Psi}(x, y) = (X(x, y), Y(x, y)).$$

The identity $\widetilde{\Psi}^*\omega_0 = \omega_0$ is equivalent to saying that

$$\eta = \sum_{j=1}^{n} (y_j - Y_j)dx_j + (X_j - x_j)dY_j$$

is a closed 1-form. C^1-closeness to the identity ensures that the map $(x, y) \mapsto (x, Y)$ is a diffeomorphism of $\mathbb{R}^{2n}$. Hence we may view η in (x, Y)-space and find a real-valued function $F(x, Y)$ such that $dF = \eta$. In other words

$$\begin{aligned} X - x &= D_2F(x, Y) \\ y - Y &= D_1F(x, Y) \end{aligned} \tag{9}$$

holds, and defines the map implicitly. Finally we make use of the vanishing of the center of mass to conclude that F is $\mathbb{Z}^{2n}$-periodic. Hence, by Morse theory, the function on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ induced by F has at least $2n + 1$ critical points. By (9) this means that Ψ has at least $2n + 1$ fixed points. The non-degenerate case follows similarly by the well-known estimates on the number of critical points of a Morse function in terms of sum of Betti numbers. The proof is complete.

The Arnold conjecture for standard symplectic tori was confirmed by a celebrated result due to Conley and Zehnder [8]. Their arguments make use of degree theory in infinite dimensions in order to take advantage of the fact that the classical action functional is a compact perturbation of a quadratic form of infinite index and co-index.

The generalization to arbitrary symplectic manifolds requires the notion of Hamiltonian diffeomorphism. Let (W, ω) be a symplectic manifold and let $\{\psi^t\}_{t\in[0,1]}$ be a smooth isotopy of W starting at the identity $\psi^0 = id$, consisting of symplectic diffeomorphisms. Let X_t be the vector field generating ψ^t, i.e. $\frac{d}{dt}\psi^t = X_t \circ \psi^t$. Differentiating we obtain

$$0 = \frac{d}{dt}(\psi^t)^*\omega = (\psi^t)^*\mathscr{L}_{X_t}\omega = (\psi^t)^* d i_{X_t}\omega \tag{10}$$

from where we see that $i_{X_t}\omega$ is closed, for every t. A diffeomorphism ϕ of W isotopic to the identity is a *Hamiltonian diffeomorphism* of (W, ω) if it can be written as $\phi = \psi^1$ for some isotopy ψ^t as above such that $i_{X_t}\omega$ is exact for every t. The set of Hamiltonian diffeomorphisms of (W, ω) will be denoted by $\mathrm{Ham}(W, \omega)$.

Arnold Conjecture: Let (W, ω) *be a closed symplectic manifold and let* $\phi \in \mathrm{Ham}(W, \omega)$. *Then*

$$\#\mathrm{Fix}(\phi) \geq \inf\{\#\mathrm{Crit}(f) \mid f \in C^\infty(W, \mathbb{R})\}$$

and, moreover, if all fixed points of ϕ are non-degenerate then

$$\#\mathrm{Fix}(\phi) \geq \sum_i b_i(W)$$

where $b_i(W)$ denotes the rank of $H_i(W, \mathbb{Z})$.

The connection with the previously stated conjecture is that the set of exact symplectic diffeomorphisms of $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega_0)$ coincides precisely with $\mathrm{Ham}(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega_0)$.

To close the circle of ideas and conclude this introduction we need to explain why Arnold's conjecture on $(\mathbb{R}^2/\mathbb{Z}^2, \omega_0)$ implies Theorem 1.2 (in the smooth case). Consider an area and orientation preserving diffeomorphism f of $\mathbb{R}/\mathbb{Z} \times [0,1]$ that preserves boundary components. We make a simplifying assumption for the sake of exposition:

($*$) f is a rotation near the boundary.

If this assumption is dropped then the discussion below can be modified but the details would draw our attention to unimportant technical issues.

By the assumptions of Theorem 1.2 and by hypothesis ($*$), there is a lift $F(x,y) = (X,Y)$ of f to $\mathbb{R} \times [0,1]$ satisfying

$$\begin{aligned} F(x,y) &= (x + \alpha_0, y) \ \text{ if } y \sim 0 \\ F(x,y) &= (x + \alpha_1, y) \ \text{ if } y \sim 1 \end{aligned} \tag{11}$$

where $\alpha_0 \alpha_1 < 0$.

Consider $d_1, d_2 > 0$ to be fixed *a posteriori*, and set $d := d_1 + d_2$. On the thicker strip $\mathbb{R} \times [0, 2+d]$ consider the map Ψ defined by:

- If $y \in [0,1]$ then $\Psi(x,y) = F(x,y)$,
- If $y \in [1, 1+d_1]$ then $\Psi(x,y) = (x + \alpha_1, y)$,
- If $y \in [1+d_1, 2+d_1]$ then

$$\Psi(x,y) = (X(x, 2+d_1-y), 2+d_1 - Y(x, 2+d_1-y)),$$

- If $y \in [2+d_1, 2+d]$ then $\Psi(x,y) = (x+\alpha_0, y)$.

It follows that Ψ is a diffeomorphism of $\mathbb{R} \times [0, 2+d]$ which agrees with the map $(x,y) \mapsto (x+\alpha_0, y)$ near the boundary.

We extend Ψ to a map $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ in a $(2+d)$-periodic fashion in the y-variable. Then the extended Ψ is an area and orientation preserving diffeomorphism of $\mathbb{R}^2$ that commutes with the additive action of the lattice $\Gamma = \mathbb{Z} \times (2+d)\mathbb{Z}$. As such, it descends to a diffeomorphism on the torus $\mathbb{R}^2/\Gamma$.

If we write $\Psi(x,y) = (x + g(x,y), y + h(x,y))$ then the center of mass of Ψ is

$$\left(\int_{[0,1]\times[0,2+d]} g(x,y), \int_{[0,1]\times[0,2+d]} h(x,y) \right)$$

Now we wish to show that if d_1, d_2 are suitably chosen then the center of mass of Ψ vanishes. To see this we first note that the second coordinate vanishes: on $[0,1] \times ([1, 1+d_1] \cup [2+d_1, 2+d])$ the function $h(x,y)$ vanishes, and by the change of variables formula

$$\begin{aligned}
&\int_{[0,1]\times[1+d_1,2+d_1]} h(x,y)\\
&= \int_{[0,1]\times[1+d_1,2+d_1]} 2+d_1-Y(x,2+d_1-y)-y\\
&= \int_{[0,1]\times[1+d_1,2+d_1]} -(Y(x,2+d_1-y)-(2+d_1-y))\\
&= \int_{[0,1]\times[1+d_1,2+d_1]} -h(x,2+d_1-y) = -\int_{[0,1]^2} h(x,y).
\end{aligned}$$

As for the first coordinate, note that

$$\int_{[0,1]\times[1+d_1,2+d_1]} g(x,y) = \int_{[0,1]^2} g(x,y)$$

from where we get

$$\int_{[0,1]\times[0,2+d]} g(x,y) = \alpha_1 d_1 + \alpha_0 d_2 + 2\int_{[0,1]^2} g(x,y)$$

Since $\alpha_0\alpha_1 < 0$ we can choose d_1, d_2 positive in such a way that

$$\alpha_1 d_1 + \alpha_0 d_2 = -2\int_{[0,1]^2} g(x,y)$$

forcing the first coordinate of the center of mass to vanish, as desired.

Hence, the validity of Arnold's conjecture on the 2-torus will force Ψ to have at least three fixed points. By symmetry, F has at least two fixed points, as claimed by Theorem 1.2.

2. Hamiltonian Dynamics and PDEs

An important tool for understanding Hamiltonian dynamics is the underlying variational structure, which incarnates in many forms. This section is devoted to explaining the origins of the PDE methods used by Floer to explore the variational structure and attack the Arnold Conjecture. These methods are, of course, based on the seminal work of Gromov [19] where holomorphic curves were first introduced as a tool to study symplectic geometry.

We start with a quick revision of the basics on symplectic geometry. Phase spaces of Hamiltonian systems are *symplectic manifolds.* These are pairs (W,ω) consisting of a smooth manifold W and a closed 2-form ω which is *non-degenerate*: at every point, $\ker\omega$ is the trivial vector space. Such forms are called *symplectic forms.*

Symplectic manifolds are necessarily even dimensional, but in fact more is true: locally they are just copies of open subsets of the *standard symplectic vector space* $(\mathbb{R}^{2n},\omega_0)$ where

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j. \tag{12}$$

Here the coordinates on $\mathbb{R}^{2n}$ are denoted by $(x_1,\dots,x_n,y_1,\dots,y_n)$. This is the content of *Darboux's theorem.*

Corrections to the discussion on Floer's chain complex and a misuse of the expression "center of mass" have been added as erratum at the end of the volume.

Hamiltonian systems on (W,ω) are determined by (possibly time dependent) functions $H_t : W \to \mathbb{R}$ as follows. The *symplectic gradient*, also called the *Hamiltonian vector field*, of H_t is the non-autonomous vector field X_{H_t} defined by

$$\omega(X_{H_t},\cdot) = dH_t$$

for each t. The function H_t is called the *Hamiltonian*. In symplectic coordinates provided by Darboux theorem, the ODE

$$\dot{x}(t) = X_{H_t}(x(t)) \tag{13}$$

is nothing but Hamilton's equations of motion. The dynamics on (W,ω) induced by this ODE is *Hamiltonian dynamics*.

When the Hamiltonian H is time independent then its values are preserved by the flow of X_H since $dH(X_H) = \omega(X_H,X_H) = 0$. Then H can be thought of as *energy*, and the system is conservative.

The non-autonomous case. Let us assume that H_t is periodic in time, say of period 1. This imposes no loss of generality, as it turns out. Consider the loop space $\mathscr{L}$ of W, defined as a space of maps $\mathbb{R}/\mathbb{Z} \to W$ (regularity is not specified as this is an informal discussion). If ω is exact then one considers on $\mathscr{L}$ the *action functional* from classical mechanics

$$c \mapsto \int_{\mathbb{R}/\mathbb{Z}} c^*\lambda + \int_0^1 H_t(c(t))\ dt$$

where λ is a primitive of ω. Since ω might not be exact – and it will never be when W is closed – we need to make additional assumptions.

Assume that (W,ω) is aespherical in the sense that ω and $c_1(TW,\omega)$ vanish on $\pi_2(M)$. Here $c_1(TW,\omega)$ denotes the first Chern class of the symplectic vector bundle (TW,ω). For the reader not familiar with Chern classes, note that its vanishing on $\pi_2(M)$ is equivalent to the following condition: for every map $f : S^2 \to W$ the symplectic vector bundle $f^*(TW,\omega)$ over S^2 is trivial. Let $\mathscr{L}_0 \subset \mathscr{L}$ be the connected component consisting of contractible loops. We define

$$\mathcal{A}_H : \mathscr{L}_0 \to \mathbb{R} \qquad c \mapsto \int_{\mathbb{D}} v^*\omega + \int_{\mathbb{R}/\mathbb{Z}} H_t(c(t))dt$$

where $v : \mathbb{D} \to W$ is a smooth map satisfying $v(e^{i2\pi t}) = c(t)$. The aesphericity assumption guarantees that $\mathcal{A}_H(c)$ does not depend on the choice of v. The Euler-Lagrange equations are precisely Hamilton's equations of motion. Hence, finding (contractible) periodic motions amounts to finding critical points of $\mathcal{A}_H$.

Deep results in Symplectic Topology have been established by exploiting this variational structure. Sometimes one can define so-called *symplectic capacities* as special critical values of $\mathcal{A}_H$. These invariants shed new light onto the theory of Hamiltonian systems. In Gromov's seminal work [19] one finds the first symplectic capacity, so-called *Gromov width*, although the appropriate formalism was introduced later by Ekeland and Hofer who also defined many new symplectic capacities.

There are major difficulties in the analysis of $\mathcal{A}_H$. Critical points have infinite index and co-index, in any reasonable sense. Moreover, $\mathcal{A}_H$ is unbounded from above and below. If one follows any kind of anti-gradient flow of $\mathcal{A}_H$ then critical points and values are expected to be missed, and sublevel sets do not change topology as we pass critical levels.

Floer's idea is to look at the L^2 anti-gradient flow equation, even knowing that it does not define a flow. To be more concrete, consider ω-compatible and 1-periodic almost complex structures J_t on W. These are 1-parameter families of fields of endomorphisms J_t of TW such that for every t there holds $J_{t+1} = J_t$, $J_t^2 = -I$ and $\omega(\cdot, J_t\cdot)$ is a Riemannian metric. Then

$$\langle X, Y\rangle_{L^2} = \int_{\mathbb{R}/\mathbb{Z}} \omega(c(t))(X(t), J_t(c(t))Y(t))\ dt$$

defines an inner-product on vector fields along c (thought of as tangent vectors of the loop space, based at c). The L^2 anti-gradient of $\mathcal{A}_H$ at c is the vector field

$$-J_t(c(t))[\dot{c}(t) - X_{H_t}(c(t))]$$

along c. If $u = u(s,t)$ is a map $\mathbb{R}\times\mathbb{R}/\mathbb{Z} \to W$, thought of as a path of loops, then the anti-gradient flow equation is *Floer's equation*

$$u_s + J_t(u)[u_t - X_{H_t}(u)] = 0. \tag{14}$$

It is a non-linear elliptic equation, a compact perturbation of the Cauchy-Riemann equation associated to J. Its analytical properties suffice to define some kind of Morse homology of $\mathcal{A}_H$, called *Floer homology.* Note that if $u(s,t)$ solves (14) then $u(s+s_0,t)$ is also a soluation, for every s_0, but they will be declared equivalent for the purpose of counting solutions.

Let us pause and describe a simple instance of Floer's construction. By the aesphericity condition every contractible 1-periodic solution $x : \mathbb{R}/\mathbb{Z} \to W$ of (13) has a well-defined Conley-Zehnder index $\mathrm{CZ}(x) \in \mathbb{Z}$. We shall not describe the Conley-Zehnder index, referring to [4] for details; we shall only say that it is related to winding properties of the linearization of the ODE (13).

Floer assumes that contractible 1-periodic solutions of (13) are non-degenerate in the sense that 1 is not an eigenvalue of the corresponding linearized flow. This is like saying that the action functional is Morse. He considers the vector space $\mathrm{CF}_*(H)$ over $\mathbb{Z}/2\mathbb{Z}$ freely generated by the contractible 1-periodic solutions of (13), graded by the Conley-Zehnder index. It turns out that when J_t is generic – in some precise sense that we will not explain here – the following holds: if x, y are generators of $\mathrm{CF}(H)$ satisfying $\mathrm{CZ}(x) - \mathrm{CZ}(y) = 1$ then the number of equivalence classes of solutions $u(s,t)$ of Floer's equation (14) satisfying

$$\lim_{s\to-\infty} u(s,t) = x(t) \qquad \lim_{s\to+\infty} u(s,t) = y(t)$$

is finite. Denote the number of such equivalence classes of solutions by $n(x,y)$. Floer defines a degree -1 differential

$$\delta_{H,J}(x) = \sum_{\mathrm{CZ}(y)=\mathrm{CZ}(x)-1} (n(x,y) \mod 2)\ y \tag{15}$$

on $\mathrm{CF}(H)$. It turns out that the homology of the chain complex $(\mathrm{CF}_*(H), \delta_{H,J})$ is independent of the pair (H,J) and is canonically isomorphic to the singular homology $H_*(W, \mathbb{Z}/2\mathbb{Z})$. This explains why the sum of Betti numbers is a lower bound on the number of contractible 1-periodic solutions of (13), at least in the non-degenerate case.

This homology theory carries a richer structure: a filtration by action values. This fact is used to define special critical values, which in turn are used to construct invariants. Hence the variational structure of $\mathcal{A}_H$ sheds light onto the geometry of the phase space. But we are also very much interested in the reverse direction: the properties of (W,ω) might force $\mathcal{A}_H$ to have special critical points. These periodic orbits can be used to study global properties of the underlying Hamiltonian system. For the basics in Floer homology we refer to the book [4] by Audin and Damian, and the notes [33] by Salamon.

The autonomous case. This story has a beautiful analogue for autonomous systems, which is related to contact geometry and is due to Hofer [20]. Assume H is time-independent and look at a regular level $M \subset W$. There is no loss of regularity to suppose that $M = H^{-1}(0)$ since we may add a constant to H without altering the dynamics.

Assume that ω has a primitive λ. From now on we also write λ to denote the pull-back of this primitive to M by the inclusion map $M \hookrightarrow W$, with no fear of ambiguity. The crucial assumption to be made is that λ defines a *contact form* on M, i.e. ω is pointwise non-degenerate on the associated *contact structure* $\xi := \ker\lambda \subset TM$.

The action functional on loops in M is now just $\int\lambda$. Hofer considered the so-called *symplectization.* It is defined as $\mathbb{R}\times M$ equipped with the symplectic form $d(e^a\lambda)$, where a denotes the $\mathbb{R}$-component and we see λ as an $\mathbb{R}$-invariant 1-form on $\mathbb{R}\times M$ (with respect to the action of $(\mathbb{R},+)$ on the first component).

We don't insist in parametrizing the dynamics on M by X_H. Instead, we parametrize it by the unique non-vanishing multiple X_λ of X_H satisfying $\lambda(X_\lambda)=1$. It is called the *Reeb vector field* associated to λ.

In [20] Hofer considered $\mathbb{R}$-invariant almost complex structures $\tilde{J}$ on $\mathbb{R}\times M$ that send $\frac{\partial}{\partial a} \mapsto X_\lambda$, and have the property that $\omega(\cdot,\tilde{J}\cdot)$ is an inner-product on ξ. As before, we see here X_λ and ξ as $\mathbb{R}$-invariant objects on $\mathbb{R}\times M$. Such a $\tilde{J}$ is compatible with $d(e^a\lambda)$ in the sense explained before.

The L^2 gradient flow equation is now written as the Cauchy-Riemann equation $\bar\partial_{\tilde{J}} = 0$ associated to $\tilde{J}$. Its solutions are *holomorphic curves*, but here the situation differs drastically from the one dealt by Gromov in [19]: domains necessarily need to be punctured Riemann surfaces. The behavior of these curves near their ends, under a certain finite-energy condition introduced by Hofer, allowed for Morse-homological constructions, giving rise to *Contact Homology* and *Symplectic Field Theory* [9].

3. Global surfaces of section

As explained in the introduction, Poincaré constructed annulus-like global surfaces of section in the context of the PCR3BP. Another early and powerful existence result is the following.

Theorem 3.1 (Birkhoff [6])**.** *The geodesic flow on the unit tangent bundle of any positively curved Riemannian metric on S^2 admits a global surface of section.*

In fact, we can be more precise. Consider an embedded closed geodesic $\gamma(t)$ of length L in a positively curved Riemannian 2-sphere, parametrized with unit speed. Choose an ambient orientation and for each $t \in \mathbb{R}/L\mathbb{Z}$ let $n(t)$ be the unit vector at $\gamma(t)$ such that $\{\dot\gamma(t), n(t)\}$ is a positive orthonormal frame. The Birkhoff annulus associated to γ is

$$A_\gamma = \{\cos\theta\ \dot\gamma(t) + \sin\theta\ n(t) \mid t \in \mathbb{R}/L\mathbb{Z},\ \theta\in[0,\pi]\}. \tag{16}$$

This is an embedded annulus in the unit sphere bundle, and Birkhoff showed that it is a global surface of section whenever the Gaussian curvature is positive everywhere.

Birkhoff's proof is very specific to geodesic flows and does not shed much light into the general existence problem.

A very general theory of global surfaces of section exists, for arbitrary flows on 3-manifolds. It is the outcome of the work of many mathematicians during the XX century, including Schwartzman [34], Fried [16], Ghys [18] and Sullivan [35]. However, most of the statements make use of dynamical hypotheses which are very hard to be checked. This in contrast to Theorem 3.1.

Here we focus on existence statements based on the theory developed by Hofer, Wysocki and Zehnder (HWZ) since they resonate with Birkhoff's statement in the sense that their hypotheses are quite concrete and geometric.

Consider $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) and its standard symplectic form $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let $C \subset \mathbb{R}^4$ be a compact convex set with strictly convex smooth boundary $M = \partial C$: this means that the boundary is smooth and has positive sectional curvatures everywhere. Note that any Hamiltonian having M as a regular level will define a smooth flow on M without stationary points, and another choice of such Hamiltonian will change this flow only by reparametrizing it. Hence, one can talk about Hamiltonian dynamics on M up to time-parametrization.

The first and main result is the following remarkable statement.

Theorem 3.2 (HWZ [24]). *Hamiltonian dynamics on strictly convex hypersurfaces inside* $(\mathbb{R}^4, \omega_0)$ *always admit disk-like global surfaces of section.*

Theorem 3.2 is proved using holomorphic curves. Consider $M = \partial C$ the strictly convex smooth boundary of the convex body C, and the 1-form

$$\lambda_0 = \frac{1}{2} \sum_{j=1}^{2} x_j dy_j - y_j dx_j$$

which is a primitive of ω_0. We denote $\alpha = \iota^* \lambda_0$ where $\iota : M \hookrightarrow \mathbb{R}^4$ is the inclusion. We parametrize Hamiltonian dynamics on M by the unique vector field X satisfying

$$d\alpha(X, \cdot) = 0 \qquad \alpha(X) = 1$$

which is called the Reeb vector field.

The kernel $\xi = \ker \alpha$ is a contact structure. We represent the symplectization of (M, ξ) as the symplectic manifold $(\mathbb{R} \times M, d(e^a \alpha))$ where the $\mathbb{R}$-coordinate is denoted by a. We view α, X and ξ as $\mathbb{R}$-invariant objects in the symplectization. One checks that $d\alpha$ turns ξ into a symplectic vector bundle.

Choose a complex structure $J : \xi \to \xi$ such that $d\alpha(\cdot, J\cdot) > 0$. Hofer considers the almost complex structure $\widetilde{J}$ on $\mathbb{R} \times M$ defined by

$$\widetilde{J} : \partial_a \mapsto X, \qquad \widetilde{J}|_\xi = J.$$

The proof in [24] is based on the construction of finite-energy holomorphic planes in $(\mathbb{R} \times M, \widetilde{J})$. These are smooth maps

$$\widetilde{u} = (a, u) : (\mathbb{C}, i) \to (\mathbb{R} \times M, \widetilde{J})$$

satisfying

$$\bar{\partial}(\widetilde{u}) = \frac{1}{2}(d\widetilde{u} + \widetilde{J}(\widetilde{u}) \circ d\widetilde{u} \circ i) = 0. \tag{17}$$

and a finite-energy condition introduced by Hofer [20]. HWZ construct such a plane asymptotic to a periodic trajectory $x : \mathbb{R}/T\mathbb{Z} \to M$ of X in the sense that

$$\lim_{s\to+\infty} u(e^{2\pi(s+it)}) = x(Tt) \qquad \text{in } C^\infty(\mathbb{R}/\mathbb{Z}, M).$$

The trajectory x has very special properties from dynamical and contact topological points of view: its Conley-Zehnder index is equal to 3 and its self-linking number is equal to -1. Using this information, results from [22] tell us that the M-component $u : \mathbb{C} \to M$ is an embedding transverse to X. The elliptic nature of (17) and the asymptotic behavior established in [21] allow for a Fredholm theory [23], which is used by HWZ to view such a plane as one leaf of a local foliation in $M \setminus x(\mathbb{R})$. Finally, one needs compactness results for punctured holomorphic curves to obtain a foliation on the whole of $M \setminus x(\mathbb{R})$ by planes transverse to X. This transversality and the positivity of the linearized dynamics at x are the reason why each leaf in this foliation will be the interior of a disk-like global surface of section.

The reader might wonder where is the strict convexity of M used in HWZ's argument. The answer is: both in the construction of the first holomorphic plane and in the passage from local to global foliations. The idea is that when one of these steps fail, the compactness results will produce other holomorphic curves which are asymptotic to periodic trajectories with Conley-Zehnder index less than 3. However, HWZ show that strict convexity forces the periodic Reeb trajectories to have Conley-Zehnder index at least equal to 3, a property called *dynamical convexity* by HWZ.

At this point HWZ were able to obtain the following remarkable corollary.

Theorem 3.3 (HWZ [24]). *Strictly convex energy levels in* $(\mathbb{R}^4, \omega_0)$ *admit either two or infinitely many periodic trajectories.*

To prove this, note that the return map to the disk provided by Theorem 3.2 preserves an area form with finite total area. Brouwer's translation theorem gives a fixed point, corresponding to a second periodic trajectory. Once this fixed point is removed from the disk, we are left with a return map on an open annulus. Frank's results [13, 14] imply that there are either none or infinitely many periodic points.

In the search of finer global structures of the flow, one might ask if there are more global surfaces of section on strictly convex energy levels. One might also ask if non-convex levels admit global sections. The statements below were proved using the methods of HWZ.

Theorem 3.4 ([27, 28]). *A periodic orbit on a strictly convex energy level bounds a disk-like global surface of section if, and only if, it is unknotted and has self-linking number* -1.

Theorem 3.5 ([29, 30]). *A periodic orbit on a star-shaped energy level bounds a disk-like global surface of section if it is unknotted, its self-linking number is* -1*, its Conley-Zehnder index is at least* 3*, and it is linked to all periodic orbits with transverse rotation number equal to* 1*. Conversely, these assumptions are* C^∞*-generically necessary.*

Excellent introductions to pseudo-holomorphic curve theory in symplectizations and symplectic cobordisms can be found in the book [1] by Abbas and the book [36] by Wendl.

4. Quick introduction to the PCR3BP

We end this note by giving some details on this which is a central problem in Celestial Mechanics. This also serves as an excuse to point towards a different proof of a result from [7]; see Theorem 4.2. An excellent introduction to the PCR3BP are the books [15] by Frauenfelder and van Koert, and [17] by Geiges. The book of Frauenfelder and van Koert also serves as an introduction to holomorphic curve methods and their use in Celestial Mechanics.

At first one considers three massive bodies, their positions and masses being denoted by z_1, z_2, z_3 and m_1, m_2, m_3 respectively. According to Newton's Law of Gravitation, motion is described by the following system of second order ODEs

$$\ddot{z}_1 = m_2 \frac{z_2 - z_1}{|z_2 - z_1|^3} + m_3 \frac{z_3 - z_1}{|z_3 - z_1|^3} \tag{18}$$

$$\ddot{z}_2 = m_1 \frac{z_1 - z_2}{|z_1 - z_2|^3} + m_3 \frac{z_3 - z_2}{|z_3 - z_2|^3} \tag{19}$$

$$\ddot{z}_3 = m_1 \frac{z_1 - z_3}{|z_1 - z_3|^3} + m_2 \frac{z_2 - z_3}{|z_2 - z_3|^3} \tag{20}$$

in suitably normalized units. Setting $m_3 = 0$ in (18)-(19) one obtains the *restricted* three-body problem: the first two particles (primaries) move independently of the third particle according to the two-body problem, and the third particle (massless satellite) moves according to (20). Requiring that all particles move on a plane one obtains the *planar* problem; we make this assumption here, and from now on the z_k belong to the complex plane $\mathbb{C}$. The relative position of the primaries $\zeta = z_2 - z_1$ solves Kepler's equation $\ddot{\zeta} = -(m_1 + m_2)\zeta/|\zeta|^3$ and the adjective *circular* refers to the case when ζ describes a circle. The PCR3BP is then the problem of describing the movement of the satellite.

One can see (20) as a non-autonomous non-linear second order differential equation for $z_3(t)$, which may lead the reader to think that it is intractable. However, a well-known miracle happens: in a certain non-inertial coordinate system the equations of motion not only retain their Hamiltonian form, but the Hamiltonian function turns out to be time-independent!

More precisely, consider a inertial coordinate system where the center of mass rests at the origin. Since ζ describes circular motion, we find $\omega \neq 0$ such that

$$z_1(t) = r_1 e^{i\omega t} \qquad z_2(t) = -r_2 e^{i\omega t} \tag{21}$$

for some constants $r_1, r_2 > 0$. The condition on the center of mass reads $m_1 r_1 - m_2 r_2 = 0$, and from Kepler's equation we extract the identity

$$(r_1 + r_2)^3 \omega^2 = m_1 + m_2.$$

There is no loss of generality to assume that the angular velocity ω is positive. In the rotating coordinate system with angular velocity ω, the position $q(t)$ of the satellite relative to the second primary is defined by

$$z_3(t) = (q(t) - r_2) e^{i\omega t}$$

from where it follows, together with (20) and (21), that $q(t)$ solves

$$\ddot{q} + 2i\omega\dot{q} - \omega^2(q - r_2) = -m_1 \frac{q - r_1 - r_2}{|q - r_1 - r_2|^3} - m_2 \frac{q}{|q|^3}. \tag{22}$$

The first primary rests at the point $r_1 + r_2$ and the second primary rests at the origin.

It is convenient to introduce the mass ratio

$$\mu = \frac{m_1}{m_1 + m_2} \in [0, 1]$$

because if ω and $m_1 + m_2 > 0$ are fixed once and for all, then all constants m_1, m_2, r_1, r_2 become functions of μ. Finally we set

$$p = \dot{q} + i\omega(q - r_2)$$

and consider the Hamiltonian

$$H_\mu(q, p) = \frac{1}{2}|p|^2 + \omega \langle q - r_2, ip\rangle - \frac{m_1}{|q - r_1 - r_2|} - \frac{m_2}{|q|} \tag{23}$$

where $\langle\cdot,\cdot\rangle$ denotes the euclidean inner-product. The miracle that materializes in front of our eyes is that (22) is equivalent to Hamilton equations

$$\dot{q} = \nabla_p H_\mu \qquad\qquad \dot{p} = -\nabla_q H_\mu$$

where the surprising fact is that this is an autonomous Hamiltonian system. As is well known, the value of H_μ is preserved along trajectories:

$$\frac{d}{dt}H_\mu = \langle \nabla_q H_\mu, \dot{q}\rangle + \langle \nabla_p H_\mu, \dot{p}\rangle = \langle \nabla_q H_\mu, \nabla_p H_\mu\rangle - \langle \nabla_p H_\mu, \nabla_q H_\mu\rangle = 0.$$

This is one reason why H_μ may be called *energy*, and we can say that energy is preserved. The value of c defined by the identity

$$-\frac{1}{2}c = H_\mu \tag{24}$$

is historically called the *Jacobi constant*.

Let us understand a bit the geometry behind some of the energy levels of H_μ. The function H_μ is unbounded from above and below. Completing the squares in (23) we get

$$H_\mu(q, p) = \frac{1}{2}|\dot{q}|^2 - U_\mu(q)$$

where

$$U_\mu(q) = \frac{1}{2}|\omega(q - r_2)|^2 + \frac{m_1}{|q - r_1 - r_2|} + \frac{m_2}{|q|} \tag{25}$$

is the *effective potential*. The projection to configuration space, i.e. to the q-plane, of the sublevel set $\{H_\mu \le -c/2\}$ coincides with the superlevel set $\{U_\mu \ge c/2\}$.

If $c \sim +\infty$ then $\{U_\mu \ge c/2\}$ has three connected components: two disk-like regions around the positions 0 and $r_1 + r_2$ of the primaries, and an unbounded component given as the complement of a large disk-like region. These are the *Hill's regions*. Their boundaries are called *ovals of zero velocity* for the following reason: if the energy of a satellite inside a bounded Hill region in $\{U_\mu \ge c/2\}$ is constrained by $H_\mu \le -c/2$ then the satellite can not leave this region, and can only touch its boundary with zero velocity $\dot{q} = 0$ and only if its energy is precisely $-c/2$. In other words, the satellite does not have enough energy to "escape" the influence of the corresponding primary.

This is the situation when the Jacobi constant $-c/2$ increases from $-\infty$ until $c/2$ reaches the highest critical critical value $c_1/2$ of U_μ. The only critical point of U_μ at this level lies strictly between the primaries, and we see the two bounded Hill regions getting "glued" together. For energy $-c/2 = -c_1/2 + \epsilon$ slightly higher than $-c_1/2$ the satellite may get arbitrarily close to both primaries, i.e. now there is enough energy to escape. If the Jacobi constant decreases to $-\infty$ then $c/2$ will cross three more critical values of U_μ corresponding to further modification of the topology of the Hill regions.

Throughout the remaining of this section we stick to values $c > c_1$ and focus on the Hill region around the origin. We make normalization assumptions $\omega = m_1 + m_2 = 1$, without loss of generality. It follows that $m_1 = r_2 = \mu$, $m_2 = r_1 = 1 - \mu$ and H_μ is written as

$$H_\mu = \frac{|p|^2}{2} + \langle ip, q - \mu \rangle - \frac{\mu}{|q-1|} - \frac{1-\mu}{|q|}$$

the unique parameter being μ. Simple calculations show that $c_1 \to 3$ both when $\mu \to 0^+$ and when $\mu \to 1^-$.

The method of Levi-Civita [31], which we shall now describe, regularizes collisions with the primary at the origin. We introduce new complex coordinates u, v by

$$q = 2v^2 \qquad p = -\frac{u}{\bar{v}}.$$

This amounts to a two-to-one transformation of 4-space which is symplectic up to a constant factor. In fact,

$$\begin{aligned} dq_1 \wedge dp_1 + dq_2 \wedge dp_2 &= \text{Re } [dq \wedge d\bar{p}] \\ &= -2 \text{ Re } \left[d(v^2) \wedge d\left(\frac{\bar{u}}{v}\right) \right] \\ &= -2 \text{ Re } \left[2v dv \wedge \left(\frac{v d\bar{u} - \bar{u} dv}{v^2} \right) \right] \\ &= -4 \text{ Re } [dv \wedge d\bar{u}] \\ &= 4 \left(du_1 \wedge dv_1 + du_2 \wedge dv_2 \right). \end{aligned}$$

It follows that Hamiltonian flows on (q, p)-space correspond to Hamiltonian flows on (u, v)-space up to a constant time-reparametrization.

We then reparametrize the Hamiltonian flow of H_μ on the level $\{H_\mu = -\frac{c}{2}\}$ as the Hamiltonian flow of $K_{\mu,c} = \frac{1}{2}|q|(H_\mu + \frac{c}{2}) = |v|^2(H_\mu + \frac{c}{2})$ on the level $\{K_{\mu,c} = 0\}$. Writing $K_{\mu,c}$ in terms of u, v we get

$$K_{\mu,c}(u,v) = \frac{1}{2}|u|^2 + 2|v|^2 \langle u, iv \rangle - \mu \text{ Im}[uv] - \frac{1-\mu}{2} - \mu \frac{|v|^2}{|2v^2 - 1|} + \frac{c}{2}|v|^2 \tag{26}$$

The upshot here is that $|q - 1| = |2v^2 - 1| > 0$ as long as $c > c_1$, hence the component $\Sigma_{\mu,c}$ of $\{K_{\mu,c} = 0\}$ corresponding to our chosen Hill region is a smooth hypersurface of $\mathbb{R}^4$ where the Hamiltonian flow is smooth. At this point it is worth pointing out the following nice statement.

Theorem 4.1 (Albers, Fish, Frauenfelder, Hofer and van Koert [2]). *For every $c > 3$ there exists $\mu_0(c) \in (0, 1)$ such that if $\mu > \mu_0(c)$ then $\Sigma_{\mu,c}$ is strictly convex, in the sense that its sectional derivatives are strictly positive. In particular Theorem 3.2 applies to give disk-like global surfaces of section.*

Consider a new set of coordinates

$$\hat{u} = u \qquad \hat{v} = \sqrt{c}\, v$$

which are again symplectic up to a constant factor, and write

$$K_{\mu,c}(\hat{u},\hat{v}) = \frac{1}{2}|\hat{u}|^2 + 2\frac{|\hat{v}|^2\,\langle \hat{u}, i\hat{v}\rangle}{c\sqrt{c}} - \mu\,\frac{\mathrm{Im}[\hat{u}\hat{v}]}{\sqrt{c}} - \frac{1-\mu}{2} - \mu\frac{|\hat{v}|^2}{c|\frac{2}{c}\hat{v}^2 - 1|} + \frac{1}{2}|\hat{v}|^2 \tag{27}$$

in terms of these variables. If $c \to +\infty$ then $K_{\mu,c}(\hat{u},\hat{v})$ converges in C^∞ to

$$\frac{1}{2}|\hat{u}|^2 + \frac{1}{2}|\hat{v}|^2 - \frac{1-\mu}{2}$$

which is the Hamiltonian of two uncoupled harmonic oscillators with equal frequencies. In particular, the flow converges to a periodic flow whose trajectories are the Hopf fibers in the sphere of radius $\sqrt{1-\mu}$. In fact, the C^∞-converge is uniform when μ ranges on a compact subset of $[0,1)$. This simple observation and standard arguments can be used to prove

Theorem 4.2 (Conley [7]). *If $\mu \in (0,1)$ is fixed and the Jacobi constant is large enough then there are annulus-like global surfaces of section for the Hamiltonian flow on the component of the energy level $\{K_{\mu,c} = 0\}$ corresponding to the Hill region around the origin. The return map to these annuli satisfy the twist condition of the Poincaré-Birkhoff Theorem, in particular there are infinitely many periodic orbits for the PCR3BP in these situations.*

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