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EQUIVARIANT ALGEBRAIC KK-THEORY

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Abstract. In this paper we present results from [6] and [7] about equivariant algebraic kk-theory.

1. Kasparov's KK-theory

Kasparov's KK-theory is the major tool in *noncommutative topology*, [13]. The KK-theory of separable C^*-algebras is a common generalization both of topological K-homology and tolopological K-theory as an *additive bivariant functor* Let A and B separable C^*-algebras then a *group* $KK(A,B)$ is defined such that

$$KK_*(\mathbb{C},B) \simeq K_*^{top}(B) \qquad KK^*(A,\mathbb{C}) \simeq K^*_{hom}(A).$$

An important property of KK-theory is the so-called *Kasparov product*,

$$KK(A,B) \times KK(B,C) \to KK(A,C)$$

which is bilinear with respect to the additive group structures. Denote by C^*-Alg to the category of separable C^*-algebras. The Kasparov groups $KK(A,B)$ for $A, B \in C^*$-Alg form a morphisms sets $A \to B$ of a *category* KK. The composition in KK is given by the Kasparov product and the category KK admits a *triangulated category* structure.

There is a *canonical functor* $k : C^*\text{-Alg} \to KK$ that acts identically on objects and every *-homomorphism $f : A \to B$ is represented by an element $[f] \in KK(A,B)$. The functor $k : C^*\text{-Alg} \to KK$

- is homotopy invariant: $f_0 \sim f_1$ implies $k(f_0) = k(f_1)$.
- is C^*-stable: any corner embedding $A \to A \otimes \mathcal{K}(\ell^2\mathbb{N})$ induces an isomorphism $k(A) = k(A \otimes \mathcal{K}(\ell^2\mathbb{N}))$.
- is split-exact: for every split-extension $I \xrightarrow{f} A \xrightarrow{g} A/I$ (i.e. there exists a *-homomorpfhism $s : A/I \to A$ such that $g \circ s = \mathrm{id}$) then $k(I) \xrightarrow{k(f)} k(A) \xrightarrow{k(g)} k(A/I)$ is part of a distinguished triangle.

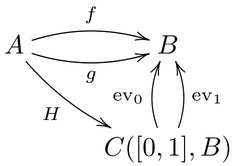
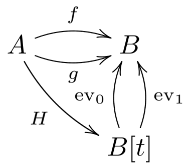
The functor $k : C^*\text{-Alg} \to KK$ is the *universal* homotopy invariant, C^*-stable and split exact functor. The main authors who worked in the previous results are: J. Cuntz [4], N. Higson [12], G. Kasparov [13] and R. Meyer [15].

2. Algebraic kk-theory

Algebraic kk-theory was introduced in [3] by G. Cortiñas and A. Thom in order to show how methods from K-theory of operator algebras can be applied in completely algebraic setting. Let ℓ a commutative ring with unit and Alg the category of ℓ-algebras (with or without unit). For each pair (A,B) of ℓ-algebras a group $\mathrm{kk}(A,B)$

is defined. A category $\mathfrak{KK}$ is obtained whose objects are ℓ-algebras and where the morphisms from A to B are the elements of the group $\mathrm{kk}(A, B)$. The category $\mathfrak{KK}$ is triangulated and there is a canonical functor $j : \mathrm{Alg} \to \mathfrak{KK}$ with universal properties. These properties are algebraic homotopy invariance, matrix invariance and excision.

We resume the results with a dictionary between Kasparov's KK-theory and algebraic kk-theory as follows:

Kasparov's KK-theory [13]	$\leftrightarrow$	**Algebraic kk-theory** [3]
bivariant K-theory on C^*-Alg		bivariant K-theory on Alg
$k : C^*\text{-Alg} \to KK$		$j : \mathrm{Alg} \to \mathfrak{KK}$
k is stable w.r.t. compact operators		j is stable w.r.t. matrices
$A \simeq_{KK} A \otimes \mathcal{K}(\ell^2(\mathbb{N}))$		$A \simeq_{\mathfrak{KK}} M_\infty(A) = \bigcup_{n\in\mathbb{N}} M_n(A)$
k is continuous homotopy invariant		j is polynomial homotopy invariant
		
$B \simeq_{KK} C([0,1], B)$		$B \simeq_{\mathfrak{KK}} B[t]$
k is split exact		j is excisive
k is universal for these properties		j is universal for these properties
$KK_*(\mathbb{C}, A) \simeq K_*^{top}(A)$		$kk_*(\ell, A) \simeq \mathrm{KH}_*(A)$

KH is Weibel's homotopy K-theory defined in [22].

Theorem 2.1. [3] *The functor* $j : \mathrm{Alg} \to \mathfrak{KK}$ *is an excisive, homotopy invariant, and* M_∞*-stable functor and it is the universal functor for these properties.*

Let $\mathcal{X}$ be a infinity set. Consider

$$M_{\mathcal{X}} := \{a : \mathcal{X} \times \mathcal{X} \to \ell : \mathrm{sopp}(a) < \infty\}.$$

Let A be an algebra, then $M_{\mathcal{X}}A := M_{\mathcal{X}} \otimes_\ell A$. The category $\mathfrak{R}_{\mathcal{X}}$ is constructed as it is the category $\mathfrak{KK}$ taking $M_{\mathcal{X}}$ instead of M_∞.

Theorem 2.2. [16] *The functor* $j : \mathrm{Alg} \to \mathfrak{R}_{\mathcal{X}}$ *is an excisive, homotopy invariant, and* $M_{\mathcal{X}}$*-stable functor and it is the universal functor for these properties.*

If $\mathcal{X} = \mathbb{N}$ both theorems are the same.

3. Equivariant algebraic kk-theory

We introduce in [7] an algebraic bivariant K-theory for the category of algebras with an action of a group G or G-algebras. For each pair (A, B) of G-algebras a group $\mathrm{kk}^G(A, B)$ is defined. A category $\mathfrak{KK}^G$ is obtained whose objects are G-algebras and where the morphisms from A to B are the elements of the group $\mathrm{kk}^G(A, B)$. The category $\mathfrak{KK}^G$ is triangulated and there is a canonical functor $j : G\text{-Alg} \to \mathfrak{KK}^G$

with universal properties. These properties are algebraic homotopy invariance, equivariant matrix invariance and excision. Let A be a G-algebra and

$$M_G := \{a : G \times G \to \ell : \mathrm{sopp}(a) < \infty\}.$$

Consider in $M_G \otimes A$ the following action of G

$$g \cdot (e_{s,t} \otimes a) = e_{gs,gt} \otimes g \cdot a$$

A *G-stable* functor identifies any G-algebra A with $M_G \otimes A$. We consider this notion as a equivariant version of the matrix invariance.

An equivariant version of the dictionary stated above is the following:

Equivariant Kasparov's KK-theory [13]	$\leftrightarrow$	**Equivariant algebraic kk-theory** [7]
bivariant K-theory on G-C^*-Alg		bivariant K-theory on G-Alg
$k : G\text{-}C^*\text{-Alg} \to KK^G$		$j : G\text{-Alg} \to \mathfrak{KK}^G$
k is stable w.r.t. compact operators		j is G-stable
$A \simeq_{KK^G} A \otimes \mathcal{K}(\ell^2(G \times \mathbb{N}))$		$A \simeq_{\mathfrak{KK}^G} M_\infty M_G(A)$
k is continuous homotopy invariant		j is polynomial homotopy invariant
$B \simeq_{KK^G} C([0,1], B)$		$B \simeq_{\mathfrak{KK}^G} B[t]$
k is split exact		j is excisive
k is universal for these properties		j is universal for these properties
G compact		G finite and $\frac{1}{\lvert G\rvert} \in \ell$
$KK_*^G(\mathbb{C}, A) \simeq K_*^{top}(A \rtimes G)$		$kk_*^G(\ell, A) \simeq KH_*(A \rtimes G)$

Theorem 3.1 ([7])**.** *The functor $j : G$-Alg $\to \mathfrak{KK}^G$ is an excisive, homotopy invariant, and G-stable functor and it is the universal functor for these properties.*

We obtain some adjoint functors at the level of bivariant kk-theory which at the level of algebras they are not. We have algebraic versions of

- Green-Julg Theorem, see [15], [11], [1], [20], [21].
- Adjointness property between Ind_H^G and Res_G^H, see [15].
- Baaj-Skandalis Duality, see [2] and [20].

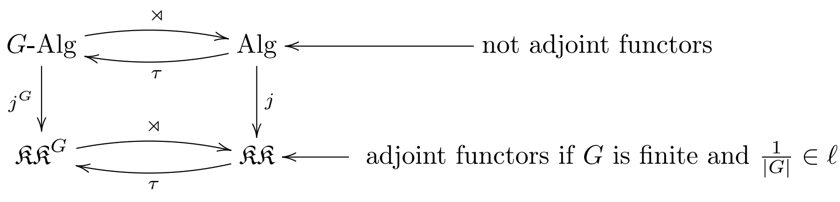
3.1. **Algebraic Green-Julg Theorem.** Consider the crossed product functor

$$\rtimes : G\text{-Alg} \to \mathrm{Alg} \qquad A \rtimes G = A \otimes \ell G \qquad (a \rtimes g)(b \rtimes h) = a[g \cdot b] \rtimes gh$$

and the trivial action functor $\tau : \mathrm{Alg} \to G$-Alg. The pair

$$G\text{-Alg} \underset{\tau}{\overset{\rtimes}{\rightleftarrows}} \mathrm{Alg}$$

can be extended to



Theorem 3.2. [7] *Let G be a finite group of n elements such that $\frac{1}{n} \in \ell$. Let B be a G-algebra and A an algebra. There is an isomorphism*

$$\psi_{GJ} : kk^G(A^\tau, B) \to kk(A, B \rtimes G)$$

We give an example to show that the adjointness between of τ and $\rtimes$ fails to hold at the algebra level. Let $G = \mathbb{Z}_2 = \{1, \sigma\}$, $A = \ell$ and $B = (\ell G)^*$ the dual algebra of ℓG with the regular action. Note $\hom_{G\text{-Alg}}(A^\tau, B)$ has two elements only:

$$\varphi_i : \ell \to (\ell G)^* \qquad \varphi_0(1) = 0 \qquad \varphi_1(1) = \chi_1 + \chi_\sigma$$

One the other hand $\hom_{\text{Alg}}(A, B \rtimes G) = \hom_{\text{Alg}}(\ell, (\ell G)^* \rtimes G)$ has at least as many elements as ℓ. For each $\lambda \in \ell$ we can define

$$\varphi_\lambda : \ell \to (\ell G)^* \rtimes G \quad \varphi_\lambda(1) = \chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma) \quad \lambda \in \ell$$

Note φ_λ is an algebra morphism because $\chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma)$ is an idempotent element.

Corollary 3.3. [7] *Let G be a finite group of n elements such that $\frac{1}{n} \in \ell$. Then*

$$kk^G(\ell, B) \simeq \mathrm{KH}(B \rtimes G) \qquad kk^G(\ell, \ell) \simeq \mathrm{KH}(\ell G)$$

3.2. **Adjointness between Ind_H^G and Res_G^H.** Let H be a subgroup of G and A an H-algebra. Define

- $A^{(G)} = \{\alpha : G \to A : \alpha \text{ is a function with finite support}\}$
- $\mathrm{Ind}_H^G(A) = \{\alpha \in A^{(G)} : \alpha(s) = h \cdot \alpha(sh) \quad \forall h \in H, s \in G\}$
- $(g \cdot \alpha)(s) = \alpha(g^{-1}s)$

The functors

$$G\text{-Alg} \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_G^H}{\rightleftarrows}} H\text{-Alg} \longleftarrow \mathrm{Ind}_H^G \text{ is NOT left adjoint to } \mathrm{Res}_G^H$$

can be extended to

$$\mathfrak{KK}^G \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_G^H}{\rightleftarrows}} \mathfrak{KK}^H \longleftarrow \mathrm{Ind}_H^G \text{ is a left adjoint to } \mathrm{Res}_G^H$$

Theorem 3.4. [7] *Let G be a group, H a subgroup of G, B an H-algebra and A a G-algebra. Then there is an isomorphism*

$$\psi_{IR} : kk^G(\mathrm{Ind}_H^G(B), A) \to kk^H(B, \mathrm{Res}_G^H(A))$$

Let us show that this adjunction is not true at the level of algebras. Consider $G = \mathbb{Z}_2 = \{1, \sigma\}$, $H = \{1\}$, $\ell = \mathbb{Q}$, A be any $\mathbb{Q}$-algebra and A^τ is the algebra A with the trivial action of G. Let us compute $\hom_{G\text{-Alg}}(\mathrm{Ind}_H^G(\ell), A^\tau)$ and $\hom_{\text{Alg}}(\ell, A)$. We obtain

$$\mathrm{Ind}_H^G(\ell) = \mathbb{Q}^{\mathbb{Z}_2} = \{a\chi_1 + b\chi_\sigma : a, b \in \mathbb{Q}\}$$

If $\alpha \in \hom_{G\text{-Alg}}(\mathbb{Q}^{\mathbb{Z}_2}, A^\tau)$ then $\alpha(\chi_1) = p$ with p an idempotent element of A. We obtain

$$\alpha(\chi_\sigma) = \alpha(\sigma \cdot \chi_1) = \sigma \cdot \alpha(\chi_1) = \sigma \cdot p = p$$

and

$$0 = \alpha(\chi_1 \chi_\sigma) = \alpha(\chi_1)\alpha(\chi_\sigma) = p^2 = p$$

Finally $\hom_{G\text{-Alg}}(\mathbb{Q}^{\mathbb{Z}_2}, A^\tau) = 0$. On the other hand $\hom_{\text{Alg}}(\ell, A)$ has at least as many elements as idempotents of A.

Corollary 3.5. [7]

- $kk^G(\ell^{(G/H)}, A) \simeq kk^H(\ell, \mathrm{Res}_G^H(A))$.
- *If* H *is finite then* $kk^G(\ell^{(G/H)}, A) \simeq \mathrm{KH}(A \rtimes H)$.
- $kk^G(\ell^{(G)}, A) \simeq \mathrm{KH}(A)$.

3.3. **Algebraic Baaj-Skandalis duality.** A dual notion of G-algebra is the concept of G-graded algebra. A *G-graduation* on an algebra A is a decomposition on submodules

$$A = \bigoplus_{s \in G} A_s \quad A_s A_t \subseteq A_{st} \quad \forall s, t \in G$$

Let A be a G-algebra. Then

$$A \rtimes G = \bigoplus_{s \in G} A \rtimes s \qquad \text{and } (A \rtimes s)(A \rtimes t) \subset A \rtimes st$$

thus $A \rtimes G$ is a G-graded algebra. Let B be a G-graded algebra. Let $G \hat{\ltimes} B$ be the algebra which as a module is $\ell^{(G)} \otimes B$ and the product is the following

$$(\chi_g \ltimes a)(\chi_h \ltimes b) := \chi_g \ltimes a_{g^{-1}h} b. \tag{3.6}$$

Here b_g is the homogeneous element associated to g in the decomposition

$$b = \sum_{g \in G} b_g.$$

One checks that the product (3.6) is associative and the action of G

$$s \cdot \chi_g \ltimes a = \chi_{sg} \ltimes a$$

makes it into a G-algebra.

The functors

$$G\text{-Alg} \underset{\hat{\ltimes}}{\overset{\rtimes}{\rightleftarrows}} G_{gr}\text{-Alg}$$

can be extended to

$$\begin{array}{ccccc}
G\text{-Alg} & \underset{\hat{\ltimes}}{\overset{\rtimes}{\rightleftarrows}} & G_{gr}\text{-Alg} & \longleftarrow & \text{not an equivalence} \\
{\scriptstyle j^G}\downarrow & & \downarrow{\scriptstyle \hat{j}^G} & & \\
\mathfrak{KK}^G & \underset{\hat{\ltimes}}{\overset{\rtimes}{\rightleftarrows}} & \hat{\mathfrak{KK}}^G & \longleftarrow & \text{an equivalence}
\end{array}$$

4. Algebraic quantum kk-theory

In [6] we introduce an equivariant algebraic kk-theory for $\mathcal{G}$-modules algebras where $\mathcal{G}$ is an algebraic quantum group.

4.1. **Van Daele's algebraic quantum groups.** Let $\ell = \mathbb{C}$ and $(\mathcal{G}, \Delta, \varphi)$ be an algebraic quantum group (see [5],[18] and [19]), in other words a regular multiplier Hopf algebra with invariants. That means, $(\mathcal{G}, \Delta)$ is a *multiplier Hopf algebra*:

- $\mathcal{G}$ associative algebra over $\mathbb{C}$ with non-degenerate product.
- $M(\mathcal{G})$ multiplier algebra of $\mathcal{G}$: $(\rho_1, \rho_2) \in M(\mathcal{G})$ if
  - $\rho_i : \mathcal{G} \to \mathcal{G}$ is a linear map $(i = 1, 2)$
  - $\rho_1(hk) = \rho_1(h)k \quad \rho_2(hk) = h\rho_2(k) \quad \rho_2(h)k = h\rho_1(k) \quad \forall h, k \in \mathcal{G}$

The multiplication in $M(\mathcal{G})$ is defined as follows

$$(\rho_1, \rho_2)(\tilde{\rho}_1, \tilde{\rho}_2) = (\rho_1\tilde{\rho}_1, \tilde{\rho}_2\rho_2)$$

- An homomorphism $\Delta : \mathcal{G} \to M(\mathcal{G} \otimes \mathcal{G})$ is a comultiplication if
  - $\Delta(h)(1 \otimes k) \in \mathcal{G} \otimes \mathcal{G} \qquad (h \otimes 1)\Delta(k) \in \mathcal{G} \otimes \mathcal{G} \quad \forall h, k \in \mathcal{G}$
  - The coassociativity property is satisfied:

$$(h \otimes 1 \otimes 1)(\Delta \otimes \mathrm{id}_{\mathcal{G}})(\Delta(k)(1 \otimes r)) = (\mathrm{id}_{\mathcal{G}} \otimes \Delta)((h \otimes 1)\Delta(k))(1 \otimes 1 \otimes r) \quad \forall h, k, r \in \mathcal{G}$$

- The maps $T_i : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ such that

$$T_1(h \otimes k) = \Delta(h)(1 \otimes k) \quad \text{and} \quad T_2(h \otimes k) = (h \otimes 1)\Delta(k)$$

  are bijective.

If $(\mathcal{G}, \Delta)$ is a multiplier Hopf algebra there is a unique homomorphism $\epsilon : \mathcal{G} \to \mathbb{C}$, called counit, such that

$$(\epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Delta(h)(1 \otimes k)) = hk \qquad (\mathrm{id}_{\mathcal{G}} \otimes \epsilon)((h \otimes 1)\Delta(k)) = hk \quad \forall h, k \in \mathcal{G}.$$

There is also a unique anti-homomorphism $S : \mathcal{G} \to M(\mathcal{G})$, called antipode, such that

$$m(S \otimes \mathrm{id}_{\mathcal{G}})(\Delta(h)(1 \otimes k)) = \epsilon(h)k \qquad m(\mathrm{id}_{\mathcal{G}} \otimes S)((h \otimes 1)\Delta(k)) = \epsilon(k)h \quad \forall h, k \in \mathcal{G}$$

here m is the multiplication map. $(\mathcal{G}, \Delta)$ is a *regular multiplier Hopf algebra* if $S(\mathcal{G}) \subseteq \mathcal{G}$ and S is invertible. There is a natural embedding $\iota_{\mathcal{G}} : \mathcal{G} \to M(\mathcal{G})$ which is an homomorphism

$$h \mapsto (L_h, R_h) \qquad L_h(k) = hk \qquad R_h(k) = kh$$

Moreover $\rho h \in \mathcal{G}$ and $h\rho \in \mathcal{G}$ for all $h \in \mathcal{G}$ and $\rho \in M(\mathcal{G})$,

$$\rho h = (L_{\rho_1(h)}, R_{\rho_1(h)}) \qquad h\rho = (L_{\rho_2(h)}, R_{\rho_2(h)})$$

We write $\rho h = \rho_1(h)$ and $h\rho = \rho_2(h)$.

A *right invariant functional* on $\mathcal{G}$ is a non-zero linear map $\psi : \mathcal{G} \to \mathbb{C}$ such that

$$(\psi \otimes \mathrm{id}_{\mathcal{G}})\Delta(h) = \psi(h)1$$

Here $(\psi \otimes \mathrm{id}_{\mathcal{G}})\Delta(h)$ denotes the element $\rho \in M(\mathcal{G})$ such that

$$\rho k = (\psi \otimes \mathrm{id}_{\mathcal{G}})(\Delta(h)(1 \otimes k)) \qquad k\rho = (\psi \otimes \mathrm{id}_{\mathcal{G}})((1 \otimes k)\Delta(h))$$

Similarly, a *left invariant functional* on $\mathcal{G}$ is a non-zero linear map $\varphi : \mathcal{G} \to \mathbb{C}$ such that

$$(\mathrm{id}_{\mathcal{G}} \otimes \varphi)\Delta(h) = \varphi(h)1.$$

Invariant functionals do not always exist. If φ is a left invariant functional on $\mathcal{G}$ then it is unique up to scalar multiplication and $\psi = \varphi \circ S$ is a right invariant functional.

The dual of $(\mathcal{G}, \Delta)$ is $(\hat{\mathcal{G}}, \hat{\Delta})$:

- The elements of $\hat{\mathcal{G}}$ are the linear functionals of the form $\varphi(h\cdot)$

$$\hat{\mathcal{G}} = \{\xi_h : \mathcal{G} \to \mathbb{C} : \xi_h(x) = \varphi(hx)\}$$

  The elements of $\hat{\mathcal{G}}$ can also be written as $\varphi(\cdot h)$, $\psi(h\cdot)$, $\psi(\cdot h)$.
- The product on $\hat{\mathcal{G}}$ is defined as follows

$$(\xi_h \cdot \xi_k)(x) = (\varphi \otimes \varphi)(\Delta(x)(h \otimes k))$$

- The coproduct $\hat{\Delta} : \hat{\mathcal{G}} \to M(\hat{\mathcal{G}} \otimes \hat{\mathcal{G}})$ is defined by defining the elements $\hat{\Delta}(\xi_1)(1 \otimes \xi_2)$ and $(\xi_1 \otimes 1)\hat{\Delta}(\xi_2)$ in $\hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ as follows

$$((\xi_1 \otimes 1)\hat{\Delta}(\xi_2))(h \otimes k) = (\xi_1 \otimes \xi_2)(\Delta(h)(1 \otimes k))$$

$$(\hat{\Delta}(\xi_1)(1 \otimes \xi_2))(h \otimes k) = (\xi_1 \otimes \xi_2)((h \otimes 1)(\Delta(k))$$

- $(\hat{\mathcal{G}}, \hat{\Delta})$ is isomorphic to $(\mathcal{G}, \Delta)$ as algebraic quantum group

4.2. **Examples.**

- $\mathcal{G} = \mathbb{C}G$ with the usual Hopf algebra structure.

$$\varphi = \psi = \chi_e : G \to \mathbb{C} \quad \chi_e(h) = \begin{cases} 1 & e = h \\ 0 & e \neq h \end{cases}$$

$\boxed{\mathcal{G} \text{ is compact type}}$ because $1 \in \mathcal{G}$.

- $\mathcal{G} = \mathbb{C}\hat{G} = \{\sum_{g \in G} a_g \chi_g : a_g \in \mathbb{C}\ a_g \neq 0$ for a finite amount of $g\}$

$$\chi_g \chi_h = \begin{cases} \chi_g & g = h \\ 0 & g \neq h \end{cases}$$

$$\Delta : \mathcal{G} \to M(\mathcal{G} \otimes \mathcal{G})$$

$$\Delta(\chi_g) = \sum_{t \in G} \chi_{gt^{-1}} \otimes \chi_t$$

The integral is $\varphi = \psi : \mathcal{G} \to \mathbb{C} \qquad \varphi(\chi_h) = \psi(\chi_h) = 1$

$\boxed{\mathcal{G} \text{ is discrete type}}$ because exists $k \in \mathcal{G}$ suche that $xk = \epsilon(x)k$.

- $\mathcal{G} = \mathcal{H}$ a finite dimensional Hopf algebra.

$\boxed{\mathcal{G} \text{ is compact and discrete}}$

4.3. **The algebra $\hat{\mathcal{A}}(\mathcal{G})$.** Let $(\mathcal{G}, \Delta)$ be an algebraic quantum group and A be a $\mathcal{G}$-module algebra.

$$\boxed{\hat{\mathcal{A}}(\mathcal{G}) := \mathcal{G} \underset{\text{ev}}{\otimes} \hat{\mathcal{G}}} \qquad (g \otimes f)(\tilde{g} \otimes \tilde{f}) = gf(\tilde{g}) \otimes \tilde{f}$$

$$t \cdot (g \otimes f) = \sum t_{(1)} \cdot g \otimes t_{(2)} \cdot f$$

$$(t \cdot f)(g) = f(S(t)g)$$

$$\boxed{\hat{\mathcal{A}}(\mathcal{G}) \otimes A} \qquad (g \otimes f \otimes a)(\tilde{g} \otimes \tilde{f} \otimes \tilde{a}) = gf(\tilde{g}) \otimes \tilde{f} \otimes a\tilde{a}$$

$$t \cdot (g \otimes f \otimes a) = \sum t_{(1)} \cdot g \otimes t_{(3)} \cdot f \otimes t_{(2)} \cdot a$$

A functor $F : \mathcal{G}\text{-Alg} \to \mathcal{D}$ is $\mathcal{G}$-*stable* if $F(\iota_1)$ and $F(\iota_2)$ are isomorphism where

$$\iota_2 : A \to \begin{pmatrix} \hat{\mathcal{A}}(\mathcal{G}) \otimes A & 0 \\ 0 & A \end{pmatrix} \leftarrow \hat{\mathcal{A}}(\mathcal{G}) \otimes A : \iota_1$$

are corner inclusions. In [6] we define a strong and a weak stabilization. The previous definition correspond to the weak stabilization of [6], here we omit the word weak.

4.4. **Algebraic quantum kk-theory.** The following theorem allow us to define a bivariant K-theory on the category of $\mathcal{G}$-module algebras.

Theorem 4.1. [6] *Let $\mathcal{X}$ be a set such that* $\operatorname{card}(\mathcal{X}) = \mathbb{N} \times \dim_{\mathbb{C}}(\mathcal{G})$. *Let $F : \mathcal{G}\text{-}\mathrm{Alg} \to \mathcal{D}$ be a $M_{\mathcal{X}}$-stable functor. The functor*

$$\hat{F} : \mathcal{G}\text{-}\mathrm{Alg} \to \mathcal{D} \qquad A \mapsto F(\hat{\mathcal{A}}(\mathcal{G}) \otimes A)$$

is $\mathcal{G}$-stable.

Theorem 4.2. [6] *The functor $j^{\mathcal{G}} : \mathcal{G}\text{-}\mathrm{Alg} \to \mathfrak{KK}^{\mathcal{G}}$ is an excisive, homotopy invariant, and $\mathcal{G}$-stable functor. Moreover, it is the universal functor for these properties.*

4.5. **Green-Julg theorem.** Consider the smash product functor

$\# : \mathcal{G}\text{-}\mathrm{Alg} \to \mathrm{Alg}$ such that $A\#\mathcal{G} = A \otimes \mathcal{G}$ and $(a\#g)(b\#k) = \sum a(g_{(1)} \cdot b)\#g_{(2)}k$

and the trivial action functor $\tau : \mathrm{Alg} \to \mathcal{G}\text{-}\mathrm{Alg}$. The pair

$$\mathcal{G}\text{-}\mathrm{Alg} \underset{\tau}{\overset{\#}{\rightleftarrows}} \mathrm{Alg}$$

can be extended to

$$\begin{array}{ccccl} \mathcal{G}\text{-}\mathrm{Alg} & \underset{\tau}{\overset{\#}{\rightleftarrows}} & \mathrm{Alg} & \longleftarrow & \text{not adjoint functors} \\ {\scriptstyle j^{G}}\downarrow & & \downarrow{\scriptstyle j} & & \\ \mathfrak{KK}^{\mathcal{G}} & \underset{\tau}{\overset{\#}{\rightleftarrows}} & \mathfrak{KK} & \longleftarrow & \text{adjoint functors if } \mathcal{G} \text{ is a semisimple Hopf algebra.} \end{array}$$

Theorem 4.3. [6] *Let $\mathcal{H}$ be a semisimple Hopf algebra and A be an $\mathcal{H}$-module algebra then*

$$kk^{\mathcal{H}}(\mathbb{C}, A) \simeq \mathrm{KH}(A\#\mathcal{H})$$

4.6. **Baaj-Skandalis duality.** In the context of algebraic quantum groups we have a good framework for group duality and we obtain a similar theorem of Baaj-Skandalis duality. Let $\mathcal{G}$ be an algebraic quantum group.

- $\hat{\mathcal{G}}$ is a $\mathcal{G}$-module: $(g \rightharpoonup f)(k) = f(kg)$
- $\mathcal{G}$ is a $\hat{\mathcal{G}}$-module: $f \rightharpoonup g = \sum f(g_{(2)})g_{(1)}$

Theorem 4.4. [5] *Let A be a $\mathcal{G}$-module algebra then*

$$(A\#\mathcal{G})\#\hat{\mathcal{G}} \simeq \hat{\mathcal{A}}(\mathcal{G}) \otimes A$$

Theorem 4.5. [6] *The functors $\#\mathcal{G} : \mathfrak{KK}^{\mathcal{G}} \to \mathfrak{KK}^{\hat{\mathcal{G}}}$ and $\#\hat{\mathcal{G}} : \mathfrak{KK}^{\hat{\mathcal{G}}} \to \mathfrak{KK}^{\mathcal{G}}$ are equivalences and $kk^{\mathcal{G}}(A, B) \simeq kk^{\hat{\mathcal{G}}}(A\#\mathcal{G}, B\#\mathcal{G})$ where A, B are $\mathcal{G}$-module algebras.*

5. Conclusion

The equivariant algebraic kk-theory introduced in [7] is a bivariant K-theory in the category of algebras with an action of a group G. The equivariant algebraic kk-theory introduced in [6] is a bivariant K-theory in the category of $\mathcal{G}$-module algebras where $\mathcal{G}$ is an algebraic quantum group. We resume properties and adjointness theorems of each case in the following table:

	G group [7]	$\mathcal{G}$ algebraic quantum group [6]
equivariant $\mathfrak{KK}$ stability	$\mathfrak{KK}^G$ $\boxed{B \simeq_{\mathfrak{KK}^G} M_G B}$ $\mathfrak{KK}^G \underset{\tau}{\overset{\rtimes}{\rightleftarrows}} \mathfrak{KK}$	$\mathfrak{KK}^{\mathcal{G}}$ $\boxed{B \simeq_{\mathfrak{KK}^{\mathcal{G}}} \hat{\mathcal{A}}(\mathcal{G}) \otimes B}$ $\mathfrak{KK}^{\mathcal{G}} \underset{\tau}{\overset{\#}{\rightleftarrows}} \mathfrak{KK}$
Green-Julg Theorem	adjoints functors G is finite, $\frac{1}{\lvert G\rvert} \in \ell$ $\boxed{kk_*^G(\ell, A) \simeq KH_*(A \rtimes G)}$	adjoint functors if $\mathcal{G} = \mathcal{H}$ semisimple Hopf algebra $\boxed{kk_*^{\mathcal{H}}(\ell, A) \simeq KH_*(A\#\mathcal{H})}$
Ind-Res	$\mathfrak{KK}^G \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_G^H}{\rightleftarrows}} \mathfrak{KK}^H$ adjoint functors	$\boxed{?}$
Imprimitivity	$\boxed{\mathrm{Ind}_H^G(B) \rtimes G \simeq_{\mathfrak{KK}} B \rtimes H}$	$\boxed{?}$
Baaj-Skandalis duality	$\mathfrak{KK}^G \underset{\hat{\rtimes}}{\overset{\rtimes}{\rightleftarrows}} \hat{\mathfrak{KK}}^G$ equivalences	$\mathfrak{KK}^{\mathcal{G}} \underset{\#}{\overset{\#}{\rightleftarrows}} \mathfrak{KK}^{\hat{\mathcal{G}}}$ equivalences

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